

# Concerning the $L^p$ Norm of Spectral Clusters for Second-Order Elliptic Operators on Compact Manifolds

CHRISTOPHER D. SOGGE\*

*Department of Mathematics,  
University of Chicago,  
Chicago, Illinois 60637*

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In this paper we study the  $(L^p, L^2)$  mapping properties of a spectral projection operator for Riemannian manifolds. This operator is a generalization of the harmonic projection operator for spherical harmonics on  $S^n$  (see C. D. Sogge, *Duke Math. J.* **53** (1986), 43–65). Among other things, we generalize the  $L^2$  restriction theorems of C. Fefferman, E. M. Stein, and P. Tomas (*Bull. Amer. Math. Soc.* **81** (1975), 477–478) for the Fourier transform in  $R^n$  to the setting of Riemannian manifolds. We obtain these results for the spectral projection operator as a corollary of a certain “Sobolev inequality” involving  $\Delta + \tau^2$  for large  $\tau$ . This Sobolev inequality generalizes certain results for  $R^n$  of C. Kenig, A. Ruiz, and the author (*Duke Math. J.* **55** (1987), 329–347). The main tools in the proof of the Sobolev inequalities for Riemannian manifolds are the Hadamard parametrix (cf. L. Hörmander, *Acta Math.* **88** (1968), 341–370, and “The Analysis of Linear Partial Differential Equations,” Vol. III, Springer-Verlag, New York, 1985) and oscillatory integral theorems of L. Carleson and P. Sjölin (*Studia Math.* **44** (1972), 287–299) and Stein (*Ann. Math. Stud.* **112** (1986), 307–357). © 1988 Academic Press, Inc.

## 1. INTRODUCTION

Let  $M$  be a smooth connected compact manifold without boundary of dimension  $\geq 2$  and  $P$  a second-order elliptic operator on  $M$  with smooth coefficients. Assume also that  $P$  is self-adjoint with respect to a smooth positive density  $d\mu$  and that the eigenvalues,  $\lambda_j, j = 0, 1, 2, \dots$ , are ordered so that the sequence  $\{\lambda_j\}$  is monotonic. It is then well known that the  $L^2$  space with respect to  $d\mu, L^2(M)$ , can be written as the direct sum of the corresponding eigenspaces  $\mathcal{H}_j$ .

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The aim of this paper is to study the  $L^p(M) \rightarrow L^q(M)$  mapping properties of a certain projection operator associated to these eigenspaces.

Before we state things more precisely, let us review what is known in the model case where  $M = S^n$ , the Euclidean sphere, and  $P$  is the standard Laplace–Beltrami operator on  $S^n$ . In this case the distinct eigenvalues are  $-j(j+n-1)$ ,  $j=0, 1, \dots$ . Furthermore, if  $H_j$  denotes the projection operator with respect to the associated eigenspaces (i.e., the spherical harmonics of degree  $j$ ), then in a previous work the author [20] obtained sharp inequalities of the form

$$\|H_j f\|_{L^q(S^n)} \leq C j^{\sigma(p, q, n)} \|f\|_{L^p(S^n)}, \quad (1.1)$$

whenever  $1 \leq p \leq 2$  and  $q$  equals 2 or  $p'$ . As usual,  $p'$  is given by  $1/p + 1/p' = 1$ , and the exponents  $\sigma(p, q, n)$  are defined as

$$\sigma(p, p', n) = \begin{cases} n(1/p - 1/p') - 1, & 1 \leq p \leq 2(n+1)/(n+3), \\ (n-1)(2-p)/2p, & 2(n+1)/(n+3) \leq p \leq 2, \end{cases}$$

$$\sigma(p, 2, n) = \sigma(p, p', n)/2.$$

The results for spherical harmonics were proved by first establishing them for the critical exponent  $p_n = 2(n+1)/(n+3)$ , and then interpolating with respect to trivial endpoint estimates. It was possible to prove the inequalities corresponding to this critical exponent by adapting the proof of the  $L^2$  restriction theorem of Stein and Tomas [27], since rather precise estimates for the kernels associated to spherical harmonic projection operators were known (cf. [1, 20]).

In this special case, the eigenvalues repeat with a high frequency; however, for a general compact Riemannian manifold, the number of eigenvalues satisfying

$$\sqrt{|\lambda_j|} \in [k-1, k)$$

is always comparable to  $k^{n-1}$ , on account of the sharp form of the Weyl formula. (See [10].) Thus, it is not surprising that the appropriate generalization of the harmonic projection theorem (1.1) should involve projection onto a band of frequencies of width one.

Let us now state the main inequality of this paper. Let  $M$  and  $P$  be as above and let  $\chi_k$ ,  $k=1, 2, \dots$ , denote the projection operators associated to the orthogonal subspaces of  $L^2$ ,  $\sum_{j \in A_k} \mathcal{H}_j$ , where  $A_k = \{j: \sqrt{|\lambda_j|} \in [k-1, k)\}$ . As above,  $\lambda_j$  is the  $(j+1)$ st eigenvalue for  $P$ , and  $\mathcal{H}_j$  is the associated eigenspace. Thus, if  $f = \sum \varphi_j$ , where  $\varphi_j \in \mathcal{H}_j$ , then

$$\chi_k f = \sum_{j \in A_k} \varphi_j. \quad (1.2)$$

Our main result then is that, if  $1 \leq p \leq 2$  and if  $q = p'$  or  $2$ , then there is a constant  $C$  depending only on  $M$  and  $P$  for which

$$\|\chi_k f\|_{L^q(M)} \leq C k^{\sigma(p, q, n)} \|f\|_{L^p(M)}, \quad k = 1, 2, \dots \tag{1.3}$$

Here  $\sigma(p, q, n)$  is the same exponent occurring in (1.1).

It is easy to see that the inequality (1.3) contains (1.1) as a special case. Furthermore, we shall also show that if  $1 \leq p \leq p_n$ , then the results are sharp.

We should point out that if  $p \neq 2$  then the exponent  $\sigma(p, q, n)$  is always positive. If, however,  $M = T^2$  and if  $P$  is the standard Laplacian then it is a result of Cooke and Zygmund [31] that there is a constant  $C$  independent of  $j = 1, 2, \dots$  for which

$$\|f\|_{L^4(T^2)} \leq C \|f\|_{L^2(T^2)}, \quad \text{if } Pf = -(2\pi j)^2 f.$$

Comparing this inequality with the corresponding inequality in (1.3) points out the difference between the projection operators associated to individual frequencies and bands of frequencies.

In order to motivate the proof of (1.3), let us remind the reader of the connection between the special case (1.1) and certain differential inequalities. More specifically, D. Jerison [15] used the inequality (1.1) in the case where  $p = p_n$  and  $q = p'_n$  to give a new, simplified proof of the following Carleman inequality of Jerison and Kenig [16] which holds for all  $u \in C_0^\infty(\mathbb{R}^n \setminus 0)$  and  $\tau \in \mathbb{R}$  with  $\text{dist}(\tau, Z) = \frac{1}{2}$ :

$$\| |x|^{-\tau} u \|_{L^p(\mathbb{R}^{n+1}; dx/|x|^{n+1})} \leq C \| |x|^{-\tau+2} \Delta u \|_{L^p(\mathbb{R}^{n+1}; dx/|x|^{n+1})}. \tag{1.4}$$

Here  $\Delta$  denotes the Euclidean Laplace operator. Inequality (1.4) was used by Jerison and Kenig to prove a sharp unique continuation theorem for this operator.

Inequality (1.1) can be thought of as a discrete restriction lemma (cf. [20]), and, in fact, later Kenig, Ruiz, and Sogge [18] used the Euclidean restriction lemma of Stein and Tomas [27] to prove Carleman inequalities which are related to (1.4). The main inequality in the elliptic case was that if  $L(D)$  is a constant coefficient differential operator with principal part  $\Delta$  then for a sharp range of exponents  $r$  and  $s$ ,

$$\|u\|_{L^s(\mathbb{R}^n)} \leq C \|L(D)u\|_{L^r(\mathbb{R}^n)}, \tag{1.5}$$

where the constant  $C$  is independent of the lower-order terms of  $L(D)$ . The uniform inequality (1.5) was proved by using oscillatory integral theorems.

Also, Renig, Ruiz, and Sogge were able to determine a necessary condition on the exponents by noticing that

$$\lim_{\varepsilon \rightarrow 0} (\Delta + 1 + i\varepsilon)^{-1} = T,$$

where the the operator  $T$  has as its “imaginary part” the restriction operator of Stein and Tomas,

$$f \rightarrow \int_{S^{n-1}} \hat{f}(\xi) e^{ix \cdot \xi} d\xi.$$

Thus (although the argument is circular!), the differential inequalities of [18] imply the  $L^2$  restriction theorem of Stein and Tomas [27].

With these things in mind, it is not surprising that a good way to prove the “discrete restriction theorem” (1.3) is by proving a related differential inequality. This is exactly what we shall do. In fact, if  $p = p_n$  and  $\sigma = \sigma(p, 2, n)$ , then we will show that there is a constant  $C$  depending only on  $M$  and  $P$  for which

$$\|u\|_{L^p(M)} \leq Ck^{\sigma-1} \|(P \pm k^2)u\|_{L^2(M)} + Ck^\sigma \|u\|_{L^2(M)}. \quad (1.6)$$

As we shall see, the inequalities (1.3) follow easily from this one by using orthogonality, duality, and interpolation.

The strategy behind the proof of (1.6) is to first use the classical Hadamard parametrix method (see Hörmander [14]) to obtain approximate inverses for the operators  $(P \pm k^2)$ . One then obtains the appropriate bounds for the operators which arise by using an oscillatory integral theorem of Stein [24] and the Carleson–Sjölin [3] method, as in [20].

The paper is organized as follows. In Section 2 we will state the theorem involving (1.3), as well as the related differential inequalities. In this section we will also prove the sharpness results discussed above. In Section 3 we will give a brief review of the Hadamard parametrix method and collect the facts which we will need for the proof of (1.6). In Section 4 we will prove the key differential inequalities and, finally, in Section 5 we will state some open problems which seem to be related to (1.3).

In this paper we will stick to the convention that  $C$  denotes a constant which is not necessarily the same at each occurrence. Also,  $\hat{f}$  will always denote the Fourier transform for  $\mathbb{R}^n$ , and  $\partial_j = \partial/\partial x_j$ .

## 2. THE INEQUALITIES

Let us now state the following two sets of inequalities which are the main results of this paper. For reasons of exposition, we have chosen to state them separately, even though, as we shall see, they are in fact equivalent.

We shall stick to the notation introduced above. That is,  $M$  will denote a smooth connected compact manifold of dimension  $\geq 2$ , to which we associate a self-adjoint second-order elliptic operator  $P$  with smooth coefficients. Also,  $\chi_k$  will denote the projection operator associated to the band of frequencies of  $P$  which satisfy  $\sqrt{|\lambda_j|} \in [k - 1, k)$ , and  $p_n$  will denote the critical exponent  $p_n = 2(n + 1)/(n + 3)$ .

**THEOREM 2.1.** *Fix a manifold  $M$  and second-order elliptic operator  $P$  as above. Then,*

(i) if  $f \in L^p(M)$ ,  $1 \leq p \leq p_n$ ,

$$\|\chi_k f\|_{L^p(M)} \leq Ck^{n[1/p - 1/p_n] - 1} \|f\|_{L^p(M)}, \tag{2.1}$$

(ii) if  $f \in L^p(M)$ ,  $p_n \leq p \leq 2$ ,

$$\|\chi_k f\|_{L^p(M)} \leq Ck^{(n-1)(2-p)/2p} \|f\|_{L^p(M)}. \tag{2.2}$$

Here, the constant  $C$  depends only on  $M$  and  $P$ . Moreover, the inequalities (2.1) are sharp.

**THEOREM 2.2.** *Fix  $M$  and  $P$  as above. Then,*

(i) if  $f \in L^p(M)$ ,  $1 \leq p \leq p_n$ , and if  $\sigma(p, n)$  is the exponent in (2.1),

$$\|\chi_k f\|_{L^2(M)} \leq Ck^{\sigma(p, n)/2} \|f\|_{L^p(M)}, \tag{2.3}$$

(ii) if  $f \in L^p(M)$ ,  $p_n \leq p \leq 2$ , and if  $\sigma(p, n)$  is the exponent in (2.2),

$$\|\chi_k f\|_{L^2(M)} \leq Ck^{\sigma(p, n)/2} \|f\|_{L^p(M)}. \tag{2.4}$$

As above, the inequality (2.3) is sharp and  $C$  depends only on  $M$  and  $P$ .

*Remark.* It is likely that (2.2) and (2.4) are also sharp. The proof of the sharpness of (2.1) and (2.3) breaks down in this case since inequalities (2.2) and (2.4) are not strong enough to imply the sharp Sobolev inequalities involving powers of  $P$  (cf. (2.8)). In the special case of spherical harmonics on  $S^n$ , however, it is known that they are sharp, since one can write down a sequence of spherical harmonics of degree  $k$  with the appropriate Lebesgue norms (see [20]).

Before we state the key ingredients in the proofs, let us indicate why the two theorems imply one another.

First of all, fix  $1 \leq p \leq 2$ , and let  $\sigma = \sigma(p, n)$  be the corresponding

exponent in Theorem 2.1. Then since the exponent associated to  $p$  in Theorem 2.2 is  $\sigma/2$ , duality clearly gives the inequalities

$$\|\chi_k f\|_{p'} \leq Ck^{\sigma/2} \|\chi_k f\|_2 \leq (Ck^{\sigma/2})^2 \|f\|_p,$$

which of course shows that Theorem 2.2 implies Theorem 2.1.

The fact that Theorem 2.1 implies Theorem 2.2 is also easy to establish. This follows from the inequality

$$\|\chi_k f\|_2^2 \leq \|\chi_k f\|_{p'} \|f\|_p,$$

which in turn follows from orthogonality and the fact that  $\chi_k$  is a projection operator.

Having pointed these things out, it is clear that we need only prove Theorem 2.2. However, if one uses duality and the M. Riesz convexity theorem (see [25]), it is plain that Theorem 2.2 is a consequence of the trivial  $L^2 \rightarrow L^2$  estimate and the following.

**PROPOSITION 2.1.** *Let  $M$  and  $P$  be as above. Then if  $f \in L^2(M)$ ,*

$$\|\chi_k f\|_{L^{2(n+1)/(n-1)}(M)} \leq Ck^{(n-1)/2(n+1)} \|f\|_{L^2(M)}, \quad (2.5)$$

$$\|\chi_k f\|_{L^\infty(M)} \leq Ck^{(n-1)/2} \|f\|_{L^2(M)}. \quad (2.6)$$

Inequality (2.5) is much more difficult to establish. However, if  $P$  has a positive definite principal symbol, then

$$\|(P - k^2) \chi_k u\|_{L^2(M)} \leq 2k \|\chi_k u\|_{L^2(M)}.$$

Thus, (2.5) is a consequence of the following differential inequality, whose proof will be postponed until Section 4.

**LEMMA 2.1.** *Set  $\sigma = (n-1)/2(n+1)$ . Then there is an absolute constant  $C$  such that, for all  $u \in C^\infty(M)$  and  $k \in \mathbb{N}$ , one has*

$$\begin{aligned} \|u\|_{L^{2(n+1)/(n-1)}(M)} &\leq Ck^{\sigma-1} \|(P \pm k^2) u\|_{L^2(M)} \\ &\quad + Ck^\sigma \|u\|_{L^2(M)}. \end{aligned} \quad (2.7)$$

The difficulty in proving this key inequality, of course, comes from the fact that the symbol of the operators  $(P \pm k^2)$  can vanish (cf. [18]). Also, an analogous differential inequality will be used to establish (2.6); however, this will not be stated until Section 4.

Let us conclude this section by showing that the inequalities (2.1) and (2.3) of the theorems cannot be improved. By the equivalence of the two results, we need only show that (2.3) is sharp. This will follow from the

following result, which can be proved, in a straightforward manner, from results in Taylor [26, p. 308] (see also Seeley [19]).

LEMMA 2.2. *If  $1 \leq p \leq 2$  then there is a constant  $C$  so that for  $m = 1, 2, 3, \dots$  one has*

$$\left\| \sum_{k=2^m}^{2^{m+1}} \chi_k f \right\|_{L^2(M)} \leq C 2^{mn(1/p - 1/2)} \|f\|_{L^p(M)}. \tag{2.8}$$

Furthermore, this result is sharp in the sense that the above bounds cannot be replaced by  $o(2^{mn(1/p - 1/2)})$ .

To see why Lemma 2.2 implies the sharpness of (2.3), let us first give an argument showing that the estimates in (2.3) imply the lemma. This is easy, for (2.3) implies that the square of the left-hand side of (2.8) is dominated by

$$\begin{aligned} \sum_{k \in [2^m, 2^{m+1}]} \|\chi_k f\|_2^2 &\leq C \sum_{k \in [2^m, 2^{m+1}]} k^{\sigma(p, n)} \|f\|_p^2 \\ &\leq C 2^{2mn(1/p - 1/2)} \|f\|_p^2. \end{aligned}$$

This argument clearly also shows that the sharpness of (2.8) implies that the estimates in (2.3) cannot be improved, which finishes this section.

### 3. THE HADAMARD PARAMETRIX METHOD

In this section we will show how the Hadamard parametrix construction can be used to obtain a “favorable” inverse for certain second-order elliptic differential operators. We shall follow closely the presentation in Hörmander [14, pp. 30–34], making only minor changes.

Since the proof of the key differential inequality (2.7) will use local coordinates, the setting in this section will be  $\mathbb{R}^n$ ,  $n \geq 2$ . We will be working with operators  $P(x, D)$  of the form

$$P(x, D) = - \sum_{j, k} \partial_j g^{jk} \partial_k + \sum b_i \partial_i + c, \tag{3.1}$$

where the coefficients  $g^{jk}$ ,  $b_j$ , and  $c$  are smooth on an open set  $X \subset \mathbb{R}^n$  which contains the origin. Since we want  $P$  to be elliptic, we will also assume that  $(g^{jk})$  is a real positive definite matrix.

We will construct a parametrix for  $(P - z)$ , when  $z \in \mathbb{C} \setminus \mathbb{R}$ . We will be assuming that  $z$  is not real for technical reasons, but point out that this

assumption will not hurt us later on, since in (2.7) we could just as well replace  $k^2$  by  $(k+i)^2$ .

As we shall see, the construction will make use of the “Bessel potential,”  $F$ , given by

$$F(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix \cdot \xi} (|\xi|^2 - z)^{-1} d\xi. \quad (3.2)$$

This is not surprising, for if the matrix  $(g^{jk})$  were constant and if we set  $|x|_g = (\sum g_{jk} x_j x_k)^{1/2}$  (where  $(g_{jk})$  denotes the inverse matrix), it then follows that

$$\left( -\sum_{j,k} \partial_j g^{jk} \partial_j - z \right) F(|x|_g) = (\det g^{jk})^{1/2} \delta_0(x). \quad (3.3)$$

As usual,  $\delta_0$  is the Dirac distribution at the origin.

The point of the Hadamard construction is that there is a favorable generalization of (3.2) to the variable coefficient case if one chooses the coordinates well. To be more specific, if  $P(x, D)$  is as in (3.1), then one can introduce “geodesic coordinates” near 0 so that if, with an abuse of notation, in the new coordinate system we write  $P$  as in (3.1), then

$$\sum_k g_{jk}(x) x_k = \sum_k g_{jk}(0) x_k, \quad j = 1, \dots, n. \quad (3.4)$$

For a proof of this fact we refer the reader to Hörmander [14, pp. 500–501].

The last formula has a very important consequence. Namely, if we now set

$$|x|_g = \left( \sum_{j,k} g_{jk}(0) x_j x_k \right)^{1/2},$$

then

$$\sum_k g^{jk}(x) \partial_k f(|x|_g^2) = \sum_k g^{jk}(0) \partial_k f(|x|_g^2), \quad j = 1, \dots, n.$$

From this last equation and the product rule it follows that, if  $\eta \in C_0^\infty(\mathbb{R}^n)$  is supported in a small enough neighborhood of 0, and if  $P$  is as in (3.1), then

$$\begin{aligned} (P(x, D) - z) \{ \eta(x) F(|x|_g) \} &= \eta(0) (\det g^{jk}(0))^{1/2} \delta_0(x) \\ &\quad + R(x, D) F(|x|_g), \end{aligned} \quad (3.5)$$

where  $R(x, D)$  is a first-order differential operator. The crucial point is that



$R(x, D)$  does not involve second-order derivatives, but it will also be convenient to note that  $R(x, D)$  has the form

$$R(x, D) = \eta_0(x) + \sum_{j=1}^n \eta_j(x) \partial_j, \tag{3.6}$$

for appropriate  $C_0^\infty$  functions  $\eta_j$  which have the same support properties as  $\eta$  and do not depend on the parameter  $z$ .

*Remark.* We should point out that, as in Hormander [14, pp. 32–33], one could “improve” this rather crude parametrix by including terms involving inverse Fourier transforms of  $(|\xi|^2 - z)^{-\nu}$ ,  $\nu \geq 2$ . However, this “improvement” is unnecessary since, as we shall see, the remainder for the parametrix with just one term has the desired  $L^2 \rightarrow L^q$  mapping properties.

Formulae (3.5) and (3.6) would be sufficient for obtaining the  $L^2(M) \rightarrow L^\infty(M)$  estimates of Proposition 2.1; however, for the  $L^2(M) \rightarrow L^{2(n+1)/(n-1)}(M)$  estimates, we will need a generalization which involves a parameter  $y \in \mathbb{R}^n$ . This, though, follows in a straightforward manner from the construction described above (see [14, pp. 33–34]).

In fact, if  $\eta \in C_0^\infty(\mathbb{R}^n \times \mathbb{R}^n)$  is supported in a small enough neighborhood of  $(0, 0)$ , then there is a Riemannian distance function  $s(x, y)$  defined on  $\text{supp}(\eta)$  for which

$$\begin{aligned} P(x, D) \{ \eta(x, y) F(s(x, y)) \} \\ = \eta(y, y) (\det g^{jk}(y))^{1/2} \delta_{jk}(x) + R(x, y, D) F(s(x, y)), \end{aligned} \tag{3.7}$$

where, this time,  $R(x, y, D)$  is a first-order differential operator depending on the parameter  $y$  of the form

$$R(x, y, D) = \eta_0(x, y) + \sum \eta_j(x, y) \tag{3.8}$$

for the appropriate  $C_0^\infty$  functions  $\eta_j$  which have the same support properties as  $\eta$ .

The function  $s(x, y)$ , of course, depends on the changes of coordinates involved. One could write down a formula for this function (see [14, p. 34]); however, the only properties we will want to use is that  $(s(x, y))^2$  is  $C^\infty$  and that, moreover, if  $g^{jk}(0) = \delta_{jk}$ , it follows that near  $(0, 0)$ ,  $(s(x, y))^2 = |x - y|^2 + \text{higher-order terms}$ .

Let us conclude this section by showing how formulae (3.7) and (3.8) can be used to “invert” the operator  $(P - z)$ . More specifically, by taking adjoints, it follows that, if  $\eta \in C_0^\infty(\mathbb{R}^n \times \mathbb{R}^n)$  has small enough support, and if we set  $\tilde{\eta}(x) = (\det g^{jk}(x))^{1/2} \eta(x, 0)$ , then

$$\tilde{\eta}(x) u(x) = T_1 [(P(\cdot, D) - z) u](x) + (T_2 u)(x), \tag{3.9}$$

where the operators  $T_j$  can be written as

$$(T_j f)(x) = \int K_j(x, x-y) f(y) dy, \quad (3.10)$$

for kernels  $K_j$  of the form

$$K_1(x, x-y) = \eta(x, x-y) F(s(x, y)) \quad (3.11)$$

and

$$K_2(x, x-y) = \eta_0(x, x-y) F(s(x, y)) + \sum \eta_j(x, x-y) \partial_j F(s(x, y)). \quad (3.12)$$

As before, the  $C_0^\infty$  functions  $\eta_j$  have the same support properties as  $\eta$ . Formula (3.9) will of course be useful if  $\tilde{\eta}(x) = 1$  near 0.

In the next section we will use oscillatory integral theorems to show that the operators  $T_j$  have the appropriate bounds in terms of the parameters  $z = (k+i)$ . We remark that  $T_2$  is a "worse" operator since it involves derivatives of  $F$ , and that this accounts for the different exponents in the differential inequality (2.7).

#### 4. PROOFS OF THE INEQUALITIES

As above, let  $M$  be a smooth connected compact manifold of dimension  $\geq 2$  and  $P$  a self-adjoint second-order elliptic differential operator on  $M$  with smooth coefficients. Since the aim of this section will be to prove differential inequalities which imply Proposition 2.1, clearly there will be no loss of generality in assuming from now on that the principal symbol of  $P$  is positive definite. This of course will imply that  $\lim_{j \rightarrow \infty} \lambda_j = +\infty$ , where as before  $\{\lambda_j\}$  are the eigenvalues of  $P$ .

Throughout this section,  $q$  will always denote the exponent  $p'_n = 2(n+1)/(n-1)$ , and we will let  $\alpha(q) = (n-1)/2(n+1)$  and  $\alpha(\infty) = (n-1)/2$ . Finally,  $B_\varepsilon$  will denote the ball of radius  $\varepsilon$  in  $\mathbb{R}^n$ .

Having set this notation, the main goal of this section will be to prove the following two inequalities which are to be uniform in  $k = 1, 2, \dots$ :

$$\|u\|_{L^q(M)} \leq Ck^{\alpha(q)-1} \|(P - (k+i)^2)u\|_{L^2(M)} + Ck^{\alpha(q)} \|u\|_{L^2(M)}, \quad (4.1)$$

$$\|x_k u\|_{L^x(M)} \leq Ck^{\alpha(\infty)-1} \|(P - (k+i)^2)u\|_{L^2(M)} + Ck^{\alpha(\infty)} \|u\|_{L^2(M)}. \quad (4.2)$$

It is for technical reasons that it is convenient to put  $\chi_k u$  in the left-hand side of (4.2); however, this will not affect the application we have in mind. In fact, if one recalls the definition of  $\chi_k$  and uses the fact that the principal

symbol of  $P$  is positive definite, then it is easy to see from orthogonality that

$$\|(P - (k + i)^2) \chi_k u\|_{L^2(M)} \leq Ck \|u\|_{L^2(M)}.$$

On account of this last observation, it is plain that (4.1) and (4.2) imply Proposition 2.1, which, as we have shown in Section 2, implies the positive statements of Theorems 2.1 and 2.2.

Let us start out by demonstrating (4.1) since it is the more interesting and difficult inequality of the two. By using the compactness of  $M$  and local coordinates it is clear that (4.1) would be a consequence of the following “local Sobolev inequality” for  $\mathbb{R}^n$ .

LEMMA 4.1. *Let  $X$  be an open subset of  $\mathbb{R}^n$ ,  $n \geq 2$ , containing the origin. Also, let  $P(x, D)$  be a second-order differential operator on  $X$  with smooth coefficients and real positive definite principal symbol. Then for all sufficiently small  $\varepsilon > 0$ , there is a constant  $C$  depending only on  $\varepsilon$  and  $P(x, D)$  such that for  $k = 1, 2, \dots$ , one has*

$$\begin{aligned} \|u\|_{L^q(B_\varepsilon)} &\leq Ck^{\alpha(q)-1} \|(P(x, D) - (k + i)^2) u\|_{L^2(B_{2\varepsilon})} \\ &\quad + Ck^{\alpha(q)} \|u\|_{L^2(B_{2\varepsilon})}. \end{aligned} \tag{4.3}$$

To prove Lemma 4.1, we will of course want to use the parametrix constructed in Section 3. To this end, let us define the radial function  $F_k(x) = F_k(|x|)$  by

$$F_k(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix \cdot \xi} (|\xi|^2 - (k + i)^2)^{-1} d\xi. \tag{4.4}$$

Taking  $z = k + i$ , we see that this is the function  $F$  which occurred in (3.2).

Next, let us notice that we can simplify things slightly by assuming from now on that the principal part of  $P(x, D)$  is equal to  $-\Delta$  at  $x = 0$ . Also, let  $s(x, y)$  be the “distance function” which arises from the Hadamard construction. It is then not difficult to see that formulae (3.9)–(3.12) imply that Lemma 4.1 is a consequence of the following.

LEMMA 4.2. *Fix  $P(x, D)$  as above and let  $s(x, y)$  be the resulting “distance function.” It then follows that, whenever  $\eta \in C_0^\infty(\mathbb{R}^n \times \mathbb{R}^n)$  is supported in a sufficiently small neighborhood of  $(0, 0)$ , there is a constant  $C$  depending only on  $\eta$  and  $P$  such that*

$$\left\| \int \eta(x, x-y) F_k(s(x, y)) f(y) dy \right\|_{L^q(\mathbb{R}^n)} \leq C k^{2(q)-1} \|f\|_{L^2(\mathbb{R}^n)}, \quad (4.5)$$

$$\left\| \int \eta(x, x-y) \nabla_y F_k(s(x, y)) f(y) dy \right\|_{L^q(\mathbb{R}^n)} \leq C k^{2(q)} \|f\|_{L^2(\mathbb{R}^n)}. \quad (4.6)$$

To prove Lemma 4.2 we will need to know a few properties of the "Bessel potentials"  $F_k$ . These are contained in the following lemma.

LEMMA 4.3. *There is an absolute constant  $C$  such that for  $k = 1, 2, \dots$ , the following results hold:*

(i) For  $n \geq 3$ ,

$$|F_k(x)| \leq C |x|^{-(n-2)}, \quad |x| \leq 1/k,$$

for  $n = 2$ ,

$$(4.7)$$

$$|F_k(x)| \leq C \log 1/|x|, \quad |x| \leq 1/k,$$

and, for  $n \geq 2$ ,

$$|\nabla F_k(x)| \leq C |x|^{-(n-1)}, \quad |x| \leq 1/k. \quad (4.8)$$

(ii) If  $|x| \geq 1/k$  and  $n \geq 2$ ,

$$F_k(x) = k^{(n-1)/2-1} e^{-ik|x|} |x|^{-(n-1)/2} a_1(kx), \quad (4.9)$$

$$\nabla F_k(x) = k^{(n-1)/2} e^{-ik|x|} |x|^{-(n-1)/2} a_2(kx), \quad (4.10)$$

for radial  $C^\infty$  functions  $a_j$  satisfying

$$|(\partial/\partial\rho)^m a_j(\rho)| \leq C_m |\rho|^{-m}. \quad (4.11)$$

These properties are more or less well known and, moreover, can be easily proved from the definition (4.4) of  $F_k$  by using stationary phase and integration by parts (cf. [8, 18]).

The proof of (4.5) will of course make use of (4.7) and (4.9). Since the proof of (4.6) can easily be obtained by modifying the argument for (4.5) and taking into account (4.8) and (4.10), we shall only give the proof of (4.5). Also, on account of (4.7), let us assume for simplicity that  $n \geq 3$  from now on. The case where  $\dim M = 2$  can be handled by obvious modifications.

To prove (4.5) let us now introduce a bump function,  $\psi \in C_0^\infty(\mathbb{R})$ , supported in  $[\frac{1}{2}, 2]$  with the property that for  $t \geq 2$ ,  $\sum_{v=1}^\infty \psi(2^{-v}t) = 1$ . Set  $\tilde{\psi}(t) = 1 - \sum_{v=1}^\infty \psi(2^{-v}t)$ .

Also, for a given  $\eta \in C_0^\infty(\mathbb{R}^n \times \mathbb{R}^n)$  whose support is small enough so that  $s(x, y)$  is defined on  $\text{supp}(\eta)$ , let us set

$$L_0(x, y) = \tilde{\psi}(k|x-y|) \eta(x, x-y) F_k(s(x, y)),$$

and for  $v = 1, 2, \dots$ ,

$$L_v(x, y) = \psi(2^{-v}k|x-y|) \eta(x, x-y) F_k(s(x, y)).$$

Inequality (4.5) would follow then, from summing a geometric series, if we could show that there is an  $\varepsilon \in (0, \frac{1}{2})$  so that whenever  $\eta \in C_0^\infty(\mathbb{R}^n \times \mathbb{R}^n)$  has the property that  $\text{supp}(\eta) \subset \{(x, y) : |x|, |y| < \varepsilon\}$  then there is a constant depending only on  $s$  and  $\eta$  so that

$$\left\| \int L_v(x, y) f(y) dy \right\|_q \leq C(2^{-v}k)^{-1/2} k^{x(q)-1} \|f\|_2, \quad v = 0, 1, 2, \dots \quad (4.12)$$

(Note that  $L_v \equiv 0$  if  $v > \log k$ .)

If we recall that  $s(x, y)$  is a Riemannian distance function near  $(0, 0)$ , then, since  $\frac{1}{2} - 1/q < 1/n$ , it is easy to see from Young's inequality and Lemma 4.3 that, if  $\varepsilon$  is small enough, then (4.12) must hold for the special case  $v = 0$ .

To prove the assertion for  $v \geq 1$  we should recall that, near  $(0, 0)$ ,  $(s(x, y))^2$  is smooth and, moreover,  $(s(x, y))^2 = |x - y|^2 + \text{higher-order terms}$ . With this in mind, it is not hard to see that the following "oscillatory integral lemma" and Lemma 4.3 imply that, for  $v \geq 1$ , (4.12) must also hold if  $\varepsilon$  is sufficiently small (cf. [20, p. 59]).

**LEMMA 4.4.** *Let  $a(x, y)$  be a function supported in the set  $\Omega = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n : |x| < \frac{1}{4}, \frac{1}{2} < |y| < 2\}$ . Then there is a neighborhood  $\mathcal{N}$  of the function  $\phi_0(x, y) = |x - y|$  in the  $C^\infty$  topology such that if  $\phi \in \mathcal{N}$  and  $\lambda > 0$  then*

$$\left\| \int e^{i\lambda\phi(x, y)} a(x, y) f(y) dy \right\|_q \leq C\lambda^{-n/q} \|f\|_2,$$

for a constant  $C$  depending only on the size of finitely many derivatives of the function  $a$ .

In two dimensions, this result is a consequence of the so-called "main lemma of Carleson and Sjölin." In higher dimensions, for the special case where  $\phi = \phi_0$  (and slightly less general  $a$ ), the result follows from an oscillatory integral theorem of Stein [24], as was shown in [20, p. 63]. Although it was not explicitly stated there, it is clear that this slightly more general case follows from the same argument that was used for  $\phi_0$ .

These last comments finish the proof of Lemma 4.1 and, consequently, (4.1).

Therefore, to finish things, we need only prove the inequality (4.2). However, the argument for this estimate is similar to the above one, so we will only sketch the important points.

First of all, let us point out that if  $P(x, D)$  is as in Lemma 4.1, then it follows from Hölder's inequality and the above reasoning that, for all sufficiently small  $\varepsilon > 0$ , there is a constant  $C$  depending only on  $P(x, D)$  and  $\varepsilon$  for which

$$\begin{aligned} \|u\|_{L^\infty(B_\varepsilon)} &\leq Ck^{-2} \|(P(x, D) - (k+i)^2)u\|_{L^\infty(B_{2\varepsilon})} \\ &\quad + Ck^{-1} \|u\|_{L^\infty(B_{2\varepsilon})} \\ &\quad + Ck^{\alpha(\infty)-1} \|(P(x, D) - (k+i)^2)u\|_{L^2(B_{2\varepsilon})} \\ &\quad + Ck^{\alpha(\infty)} \|u\|_{L^2(B_{2\varepsilon})}. \end{aligned}$$

The first two terms in the right-hand side of the above inequality occur because for  $|x| \leq 1/k$ ,  $F_k(x) \approx |x|^{-(n-2)}$  and  $|\nabla F_k(x)| \approx |x|^{-(n-1)}$ .

Nonetheless, by the compactness of  $M$ , if now  $P$  is as in (4.2) then the above inequalities imply that

$$\begin{aligned} \|u\|_{L^\infty(M)} &\leq Ck^{-2} \|(P - (k+i)^2)u\|_{L^\infty(M)} + Ck^{\alpha(\infty)-1} \|(P - (k+i)^2)u\|_{L^2(M)} \\ &\quad + Ck^{\alpha(\infty)} \|u\|_{L^2(M)}. \end{aligned}$$

To finish the proof, we need only notice that inequality (2.8) implies that

$$\|(P - (k+i)^2)\chi_k u\|_{L^\infty(M)} \leq Ck^{n/2} \|(P - (k+i)^2)u\|_{L^2(M)}.$$

Consequently, since  $n/2 - 2 < \alpha(\infty) - 1$ , the last two inequalities imply (4.2).

## 5. OPEN PROBLEMS

Let us now state a few open problems which the above results suggest.

(i) First of all, it would be interesting to know how the above "discreet restriction theorems" would carry over to the setting of  $C^\infty$  paracompact manifolds  $M$  with smooth boundary  $\partial M$ . This case seems harder than the one treated here due to the difficulties in constructing parametrices near  $\partial M$  (cf. [14, Sect. 17.5]).

(ii) Also, it would be interesting to know what the analogues of our restriction theorems would be for higher-order elliptic operators on compact manifolds  $M$ . In principle at least the case where  $\dim M = 2$  should be

approachable, since the sharp restriction theorems for degenerate curves in  $\mathbb{R}^2$  are known (see [21]).

(iii) Finally, it would be of interest to explore the possible connection between our restriction theorems and “sharp” theorems for the Riesz means of the eigenfunction expansions associated to second-order elliptic operators  $P$  on compact manifolds  $M$ . More specifically, let us define the Riesz means of index  $\delta \geq 0$  and dilation  $R > 0$ ,  $S_R^\delta$ , as

$$S_R^\delta f = \sum (1 - (\lambda_j/R)^2)_+^\delta \varphi_j, \quad f = \sum \varphi_j.$$

The conjecture (which the Euclidean results suggests, see [6]), then, is that if  $\delta > \max(0, n|\frac{1}{2} - 1/p| - \frac{1}{2})$ , then  $S_R^\delta f \rightarrow f$  in  $L^p(M)$ . This restriction on the index is necessary (see [1]); however, in most cases it is not known that it is sufficient. In fact, for  $p = 1$ , the best known result seems to be that there is  $L^1$  convergence when  $\delta > (n - 1)$  (see Hörmander [11]), while the conjecture is that  $\delta > (n - 1)/2$  should be sufficient.

For some time, however, in the Euclidean case and in the case of spherical harmonics on  $S^n$  (see [1, 7, 20]), it has been known that, for a given  $p$ , the “appropriate” ( $L^p, L^2$ ) restriction theorem implies the sharp  $L^p$  convergence theorem for Riesz means. This therefore suggests that it may be possible to use the above results to prove the sharp  $L^p(M)$  theorems for  $p$  satisfying  $1 \leq p \leq p_n$ .

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*Note added in proof.* Since this article went to press problem (iii) was solved by the author in *Annals of Math.* **126** (1987), 439–447.

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