§5. Spectral theory

5.1 The spectrum of $-\Delta$ in the periodic case.

The simplest example of an elliptic operator exhibiting the spectral properties we are now going to discuss, is the Laplace operator on a torus. With the notation used e.g. in Mat 2MA, we consider the space $L_2(\mathbb{T}^k)$ of functions $u(x)$ on $\mathbb{R}^k$ that have period 2π in each variable x_j and are square integrable on $Q =]-\pi,\pi[^k$; the space $L_2(\mathbb{T}^k)$ is a Hilbert space when provided with the scalar product and norm

$$(f,g) = (2\pi)^{-k}\int_Q f(x)\overline{g(x)}\,dx, \quad \|f\| = \Big((2\pi)^{-k}\int_Q |f(x)|^2\,dx\Big)^{\frac{1}{2}},$$

and with the convention that functions differing on a set of Lebesgue measure zero are identified with one another. Clearly, $L_2(\mathbb{T}^k) \simeq L_2(Q,(2\pi)^{-k}dx)$. The functions $u \in L_2(\mathbb{T}^k)$ can be expanded in Fourier series with respect to the orthonormal basis $\{e_n\}_{n\in\mathbb{Z}^k}$, where $e_n = e^{in\cdot x} = e^{i(n_1x_1+\cdots+n_kx_k)}$. In fact, there is a 1–1 correspondence between the functions in $L_2(\mathbb{T}^k)$ and the elements of $\ell_2(\mathbb{Z}^k)$, where f corresponds to $\{c_n\}_{n\in\mathbb{Z}^k}$ when

$$f = \sum_{n\in\mathbb{Z}^k} c_n e_n, \text{ convergence in } L_2(\mathbb{T}^k);$$

here $c_n = c_n(f) = (f,e_n)$ for each n. The unitary mapping defined in this way is denoted F,

$$F\colon L_2(\mathbb{T}^k) \xrightarrow{\sim} \ell_2(\mathbb{Z}^k), \quad \text{with } Ff = \{c_n(f)\}_{n\in\mathbb{Z}^k}.$$

By $C^m(\mathbb{T}^k)$ we denote the (Banach) space of functions in $C^m(\mathbb{R}^k)$ that have period 2π in each variable; this space identifies with the subspace of functions in $C^m(\overline{Q})$ such that all derivatives up to order m match at the boundary in such a way that the functions extend to continuous periodic functions (see e.g. Mat 2MA IV§4.4 (1)). $C^\infty(\mathbb{T}^k)$ is the Fréchet space $\bigcap_{m\in\mathbb{N}_0} C^m(\mathbb{T}^k)$. One can also define more general spaces over $\mathbb{T}^k$, e.g. distribution spaces, where one can take $\mathcal{D}'(\mathbb{T}^k)$, the space of periodic distributions, to be the dual space of $C^\infty(\mathbb{T}^k)$ (see also Schwartz [Sc 1950]). We study the general Sobolev spaces further below.

The eigenvalues and eigenfunctions of $-\Delta$ on $\mathbb{T}^k$ are extremely easy to calculate. Since $-\Delta-\lambda$ is elliptic for any λ, all eigenfunctions must be C^∞, as they satisfy an equation $(-\Delta-\lambda)u = 0$ with right hand side in C^∞. Now, each e_n is an eigenfunction, with

$$-\Delta e_n = (n_1^2+\cdots+n_k^2)e_n,$$

i.e., the corresponding eigenvalue is $\|n\|^2$. We have hereby found all eigenvalues, and a full system of eigenfunctions (in a sense explained further below); this is seen as follows:

The operator $-\Delta$ on $C^\infty(\mathbb{T}^k)$ satisfies

$$\begin{aligned}(-\Delta u, v) &= \sum_{j=1}^{k}(D_j u, D_j v) = (u, -\Delta v), \text{ in particular}\\(-\Delta u, u) &= \sum_{j=1}^{k}\|D_j u\|^2 \geq 0,\end{aligned} \tag{5.3}$$

as is seen by integrations by part. Thus $-\Delta$ with domain $C^\infty(\mathbb{T}^k)$ is symmetric and ≥ 0. It follows that all eigenvalues are ≥ 0. Moreover, when u and v are eigenfunctions belonging to different eigenvalues λ resp. μ,

$$(\lambda - \mu)(u, v) = (-\Delta u, v) - (u, -\Delta v) = 0,$$

so since $\lambda \neq \mu$, $(u, v) = 0$. This shows that *eigenfunctions belonging to different eigenvalues are orthogonal.*

The system $\{e_n\}_{n\in\mathbb{Z}^k}$ is a *complete* orthonormal system in $L_2(\mathbb{T}^k)$, as we know from Mat 2MA. Then we have found all eigenvalues; for there cannot be an eigenfunction with eigenvalue outside the set we already have, since it would be orthogonal to all the e_n and hence zero. Moreover, for each of the eigenvalues $\|n\|^2$ that we have found, the number $d(\|n\|^2)$ of vectors e_m with $\|m\|^2 = \|n\|^2$ is finite (since the coordinates are integers with absolute value $\leq \|n\|$). The full set $V(\|n\|^2)$ of eigenvectors belonging to $\|n\|^2$, with the nullvector adjoined, is a vector space containing the e_m, hence with dimension at least $d(\|n\|^2)$. Now if the dimension were $> d(\|n\|^2)$, there would be a normal vector orthogonal to all the eigenvectors we have found, which contradicts the completeness. Thus the $d(\|n\|^2)$ vectors e_m with $\|m\|^2 = \|n\|^2$ is a system of eigenvectors spanning all the eigenvectors with eigenvalue $\|n\|^2$. (The dimension of $V(\|n\|^2)$ is called the (geometric) *multiplicity of* $\|n\|^2$; we have shown that it equals $d(\|n\|^2)$.)

Now let us take a look at the associated eigenvalue system $\{\|n\|^2\}_{n\in\mathbb{Z}^k}$. First we consider multiplicities. Clearly, all eigenvalues $\neq 0$ have multiplicity ≥ 2, since the replacement of an $n_j \neq 0$ by $-n_j$ gives the same eigenvalue. Moreover, when n_1,$n_2, \dots, n_k$ are k different integers, the number $n_1^2+\cdots+n_k^2$ is eigenvalue for all the mutually distinct functions $e^{i(\sigma(n_1)x_1+\cdots+\sigma(n_k)x_k)}$ obtained when we let σ run through all permutations of the k integers. Furthermore, different sets $\{n_1, \dots, n_k\}$ can give the same eigenvalue, as for example for $k = 3$, the sets $\{1, 1, 4\}$ and $\{0, 3, 3\}$ give the same eigenvalue 18. (This indicates some first steps in the study of the eigenvalues and their multiplicities, that is related to deep problems in number theory.) We always count the eigenvalues with multiplicities (so λ is counted N times when there is an N-dimensional space of eigenfunctions associated with it). Note that 0 is a *simple* eigenvalue, i.e. has multiplicity 1.

For an overall view of the behavior of the eigenvalues of an elliptic operator one often studies the *counting function*, indicating *the number of eigenvalues less than* t, for $t \geq 0$:

$$N(t) = \#\{ \text{ eigenvalues } \leq t \}$$

(counted with multiplicity).

It can be determined in great detail in the present case, since it equals precisely the number of points n in $\mathbb{R}^k$ with integer coordinates ("grid points") for which $\|n\| \leq t^{\frac{1}{2}}$. To get an estimate of the number of grid points $n = (n_1, \dots, n_k)$ in the closed ball $\overline{B}(0, t^{\frac{1}{2}})$, we associate with each of them the k-dimensional cube with side length 1, $C_n = [n_1, n_1+1[\, \times \cdots \times [n_k, n_k+1[$. Then the points n for which $C_n \subset \overline{B}(0, t^{\frac{1}{2}})$ (we can call them the "inner" cubes) certainly have $\|n\| \leq t^{\frac{1}{2}}$, and the points such that $C_n \cap \overline{B}(0, t^{\frac{1}{2}}+1) = \varnothing$ have $\|n\| > t^{\frac{1}{2}}$ (the "outer" cubes). The union of the "inner" cubes contains the ball $\overline{B}(0, t^{\frac{1}{2}} - k^{\frac{1}{2}})$ (when $t > k$), and the complement of the union of the "outer" cubes is contained in the ball $\overline{B}(0, t^{\frac{1}{2}} + k^{\frac{1}{2}} + 1)$. The number of grid points defining the "inner" cubes equals their collected volume, which is $\geq \omega_k (t^{\frac{1}{2}} - k^{\frac{1}{2}})^k$, where ω_k is the volume of the unit ball in $\mathbb{R}^k$, and the number of grid points defining cubes that are not "outer" equals the volume of the complement of the outer cubes, hence is $\leq \omega_k (t^{\frac{1}{2}} + k^{\frac{1}{2}} + 1)^k$. Thus

$$\omega_k (t^{\frac{1}{2}} - k^{\frac{1}{2}})^k \leq N(t) \leq \omega_k (t^{\frac{1}{2}} + k^{\frac{1}{2}} + 1)^k,$$

from which we deduce that

$$N(t) = \omega_k t^{k/2} + O(t^{(k-1)/2}), \text{ for } t \to \infty. \tag{5.6}$$

This is a first example of the famous Weyl formula for the asymptotic behavior of the counting function $N(t)$ for an elliptic problem on a compact domain.

To give another example, we can change the period intervals to be different in the different coordinate directions. Consider functions with period r_j in the coordinate x_j, $j = 1, \dots, k$ (we call such functions r-periodic; $r = (r_1, \dots, r_k)$). The functions $e_{n,r} = e^{2\pi i (r_1^{-1} n_1 x_1 + \cdots + r_k^{-1} n_k x_k)}$ (with $n = (n_1, \dots, n_k) \in \mathbb{Z}^k$) are r-periodic; and they are mutually orthogonal in

$L_2(]0,r_1[\times\cdots\times]0,r_k[)$. (Let us not bother to normalize.) They are eigenfunctions of $-\Delta$ considered on the r-periodic C^∞ functions, with eigenvalues

$$\lambda_{n,r}=(2\pi)^2\Big(\frac{n_1^2}{r_1^2}+\cdots+\frac{n_k^2}{r_k^2}\Big).$$

Again we have found all the eigenfunctions, since $\{e_{n,r}\}_{n\in\mathbb{Z}^k}$ is a *complete* orthogonal system in $L_2(]0,r_1[\times\cdots\times]0,r_k[)$ (i.e. a system of nonzero vectors that by normalization gives an orthonormal basis); one can show this e.g. by carrying the situation back to $\mathbb{T}^k$ by coordinate transformations.

Again, eigenvalues $\neq 0$ have multiplicity ≥ 2 since n_j and $-n_j$ enter in the same way; but a permuation of the n_j now usually gives a different eigenvalue when the r_j are distinct, so the eigenvalues "spread out" more on $\overline{\mathbb{R}}_+$.

The counting function $N(t)$ can here be shown to satisfy

$$N(t)=\omega_{k,r}t^{k/2}+O(t^{(k-1)/2}),$$

where $\omega_{k,r}$ is the volume of the ellipsoid

$$\omega_{k,r}=\operatorname{vol}\Big\{\,x\;\Big|\;\frac{x_1^2}{r_1^2}+\cdots+\frac{x_k^2}{r_k^2}\leq\frac{1}{(2\pi)^2}\,\Big\},$$

by geometric considerations for grid points as above.

5.2 Sobolev spaces on the torus.

For any $s\in\mathbb{R}$ we denote by $\ell_{2,s}(\mathbb{Z}^k)$ the vector space of sequences $\underline{c}=\{c_n\}_{n\in\mathbb{Z}^k}$ such that

$$\|\underline{c}\|_{\ell_{2,s}}\equiv\Big(\sum_{n\in\mathbb{Z}^k}(\langle n\rangle^s|c_n|)^2\Big)^{\frac12}<\infty;$$

it is simply the Hilbert space $L_2(\mathbb{Z}^k,\langle n\rangle^{2s}\mu)$ where μ is the counting measure. (As usual, $\langle n\rangle=(1+n_1^2+\cdots+n_k^2)^{\frac12}=(1+\|n\|^2)^{\frac12}$.) Note that $\ell_{2,0}(\mathbb{Z}^k)=\ell_2(\mathbb{Z}^k)$.

It is easily verified that the differentiation operator D^α ($|\alpha|\leq m$) acts on functions in $C^m(\mathbb{T})$ by multiplication of the n'th coefficient by $n^\alpha=n_1^{\alpha_1}\dots n_k^{\alpha_k}$:

$$c_n(D^\alpha f)=(2\pi)^{-k}\int_Q D^\alpha f(x)e^{-in\cdot x}\,dx=n^\alpha c_n(f),$$

by integrations by part. The multiplication operator M_{n^α} sending $\{c_n\}$ into $\{n^\alpha c_n\}$, with domain

$$D(M_{n^\alpha})=\{\,\{c_n\}_{n\in\mathbb{Z}^k}\in\ell_2(\mathbb{Z}^k)\mid\{n^\alpha c_n\}_{n\in\mathbb{Z}^k}\in\ell_2(\mathbb{Z}^k)\,\},\tag{5.8}$$

is selfadjoint (unbounded when $|\alpha| > 0$); cf. Mat 2MA IV§4 or [MA]. We then define a generalization of D^α by

$$\mathcal{D}_\alpha = F^{-1} M_{n^\alpha} F, \tag{5.5}$$

with domain $D(\mathcal{D}_\alpha) = \{u \in L_2(\mathbb{T}^k) \mid \{n^\alpha c_n(u)\}_{n\in\mathbb{Z}^k} \in \ell_2(\mathbb{Z}^k)\}$; it is a selfadjoint operator in $L_2(\mathbb{T}^k)$ (unbounded when $|\alpha| > 0$). Still more general versions of D^α are possible, see below.

When m is an integer ≥ 0, the Sobolev space $H^m(\mathbb{T}^k)$ is defined and described by

$$\begin{aligned} H^m(\mathbb{T}^k) &= \bigcap_{|\alpha|\leq m} D(\mathcal{D}_\alpha) \\ &= \{\, u \in L_2(\mathbb{T}^k) \mid \{n^\alpha c_n(u)\}_{n\in\mathbb{Z}^k} \in \ell_2 \text{ for } |\alpha| \leq m \,\} \\ &= \{\, u \in L_2(\mathbb{T}^k) \mid \{\langle n\rangle^m c_n(u)\}_{n\in\mathbb{Z}^k} \in \ell_2 \,\} \\ &= F^{-1}\ell_{2,m}; \end{aligned}$$

it is a Hilbert space with either of the equivalent norms

$$\|u\|_m = \Big(\sum_{|\alpha|\leq m} \|\mathcal{D}_\alpha u\|_{L_2}^2\Big)^{\frac12}, \quad \|u\|_{m,\wedge} = \|Fu\|_{\ell_{2,m}}.$$

Using this as inspiration, one can define $H^s(\mathbb{T}^k)$ also for general $s \in \mathbb{R}$ as spaces of formal objects u with Fourier series

$$u \sim \sum_{n\in\mathbb{Z}^k} c_n e_n, \text{ where } \{c_n\}_{n\in\mathbb{Z}^k} \in \ell_{2,s};$$

they are Hilbert spaces with norm

$$\|u\|_{s,\wedge} = \|\{c_n\}_{n\in\mathbb{Z}^k}\|_{\ell_{2,s}} = \|\{\langle n\rangle^s c_n\}\|_{\ell_2}.$$

For $s \geq 0$, $H^s(\mathbb{T}^k)$ is a subspace of $L_2(\mathbb{T}^k)$ (consistent with the preceding definition when s is integer); it also identifies with the domain of the operator $F^{-1}M_{\langle n\rangle^s}F$. For $s < 0$, $H^s(\mathbb{T}^k)$ can be identified with the dual space of $H^{-s}(\mathbb{T}^k)$, with respect to a duality generalizing the L_2 scalar product, just as in the case of Sobolev spaces over $\mathbb{R}^k$. (All the spaces can be viewed as subspaces of $\mathcal{D}'(\mathbb{T}^k)$.) One has of course

$$H^s(\mathbb{T}^k) \subset H^t(\mathbb{T}^k) \text{ for } s > t,$$

and Theorem 2.4 and Corollary 2.5 readily generalize to these spaces:

$$\|u\|_{\theta s+(1-\theta)t,\wedge} \le \|u\|_{s,\wedge}^{\theta}\|u\|_{t,\wedge}^{1-\theta}, \tag{5.8}$$

$$\|u\|_{r,\wedge} \le \varepsilon\|u\|_{t,\wedge} + C(\varepsilon)\|u\|_{s,\wedge}. \tag{5.9}$$

Such Sobolev spaces of periodic funtions of several variables are studied also in [A 1965] and in Bers, John and Schechter [B-J-S 1964].

One has the Sobolev imbedding theorem (cf. e.g. Mat 2MA IV§4):

$$H^s(\mathbb{T}^k) \subset C^l(\mathbb{T}^k) \text{ for } s > l + \frac{k}{2},$$

l integer ≥ 0.

We can give D^α a meaning on $H^s(\mathbb{T}^k)$ by defining it as the multiplication by n^α on the coefficients c_n; it maps $H^s(\mathbb{T}^k)$ into $H^{s-|\alpha|}(\mathbb{T}^k)$. As is often done in distribution theory, we use the name D^α again, writing

$$D^\alpha \colon H^s(\mathbb{T}^k) \to H^{s-|\alpha|}(\mathbb{T}^k), \quad s \in \mathbb{R}.$$

The multiplication operators $M_{f(n)}$ above are defined in a precise sense as operators with domain and range in ℓ_2. We shall also sometimes need the more general notation

$$\mathcal{M}_{f(n)} \colon \{c_n\} \mapsto \{f(n)c_n\}$$

with unspecified domain and range; for example

$$\mathcal{M}_{\langle n\rangle^s} \colon \{c_n\} \mapsto \{\langle n\rangle^s c_n\}.$$

($M_{\langle n\rangle^s}$ is the version of this with largest possible domain and range in ℓ_2.) The corresponding operator on the expressions $\sum c_n e_n$ will be called Λ_s, so

$$\Lambda_s \colon \sum_{n\in\mathbb{Z}^k} c_n e_n \mapsto \sum_{n\in\mathbb{Z}^k} \langle n\rangle^s c_n e_n.$$

Note that

$$\Lambda_s \text{ maps } H^t(\mathbb{T}^k) \text{ isometrically onto } H^{t-s}(\mathbb{T}^k), \tag{5.10}$$

when the norms $\|u\|_{r,\wedge}$ are used; and that Λ_{-s} is the inverse of Λ_s for any s.

Now consider the Laplacian on the torus. Since $-\Delta$ for smooth functions acts like multiplication of $c_n(u)$ by $\|n\|^2$, we can define a selfadjoint realization $A_{\mathbb{T}^k}$ of $-\Delta$ by

$$A_{\mathbb{T}^k} = F^{-1} M_{\|n\|^2} F.$$

Here the domain consists of the $u \in L_2(\mathbb{T}^k)$ with $\{\|n\|^2 c_n(u)\} \in \ell_2$, so in fact

$$D(A_{\mathbb{T}^k}) = H^2(\mathbb{T}^k).$$

This operator is an extension (one can show that it is the closure) of the symmetric operator discussed in the beginning of Section 5.1; it has the same eigenvalues and eigenfunctions $\|n\|^2$ and e_n, $n \in \mathbb{Z}^k$; and no others, this follows from the completeness of the system $\{e_n\}$, as in the analysis in the beginning of Section 5.1.

5.3 Dirichlet and Neumann problems on cubes.

We can also study the Dirichlet and Neumann problems for $-\Delta$ on a cube; here we consider $Q_0 =]0,\pi[^k$. For general domains with corners and edges, it is not nearly as simple as in the smooth case to discuss the boundary operators γ_j on Sobolev spaces and to interpret the variational realizations, but in this very special constant coefficient case on the cube, one can get around much of the difficulty by using the symmetries.

Let us first define the operators in a classical way.

Define $A_{\gamma,0}$ and $A_{\nu,0}$ as $-\Delta$ with domain, respectively,

$$D(A_{\gamma,0}) = \{\, u \in C^2(\overline{Q}_0) \mid u = 0 \text{ on } \partial Q_0 \,\},$$
$$D(A_{\nu,0}) = \{\, u \in C^2(\overline{Q}_0) \mid \tfrac{\partial u}{\partial \vec{n}} = 0 \text{ on } \partial Q_0 \,\},$$

where $\frac{\partial u}{\partial \vec{n}}$ is well-defined on the open boundary hypersurfaces where one coordinate equals 0 or π and the others lie in $]0,\pi[$ (the "faces" of the cube). It is seen by integration by parts that

$$(A_{\gamma,0}u, v) = \sum_{j=1}^{k} (D_j u, D_j v) = (u, A_{\gamma,0}v), \quad (A_{\gamma,0}u, u) \ge 0,$$
$$(A_{\nu,0}u, v) = \sum_{j=1}^{k} (D_j u, D_j v) = (u, A_{\nu,0}v), \quad (A_{\nu,0}u, u) \ge 0,$$

for u and v in $D(A_{\gamma,0})$ resp. $D(A_{\nu,0})$. (Scalar products in $L_2(Q_0)$.) Again this implies, for each of the operators, that all eigenvalues are ≥ 0, and that eigenfunctions belonging to different eigenvalues are mutually orthogonal.

For the treatment of $A_{\gamma,0}$ we now recall that $L_2(Q_0)$ has the orthonormal basis $\{f_n\}_{n\in\mathbb{N}^k}$, where

$$f_n(x) = (\tfrac{2}{\pi})^{\frac{k}{2}} \sin n_1 x_1 \cdots \sin n_k x_k. \tag{5.12}$$

(The completeness can be shown by identifying $L_2(Q_0)$ with the subspace of $L_2(\mathbb{T}^k)$ consisting of functions that are *odd* in each variable $x_1, \dots, x_k$ and using the completeness of $\{e^{in\cdot x}\}_{n\in\mathbb{Z}^k}$ there, cf. e.g. Mat 2MA V§1.4.) One verifies directly that these functions are in fact eigenfunctions for $A_{\gamma,0}$, with eigenvalues

$$\|n\|^2 = n_1^2 + \cdots + n_k^2, \quad n \in \mathbb{N}^k. \tag{5.13}$$

Since the system of functions $\{f_n\}_{n\in\mathbb{N}^k}$ is a complete orthonormal system, it is seen as in the treatment of $-\Delta$ on $\mathbb{T}^k$ that (5.13) gives all the eigenvalues, and $\{f_n\}_{n\in\mathbb{N}^k}$ spans all the eigenfunctions, in the way explained there. Each eigenvalue has finite multiplicity (that can be discussed as above), and it can again be shown by geometric considerations on grid points (now in $(\mathbb{R}_+)^k$) that the counting function satisfies

$$N(t) = 2^{-k}\omega_k t^{k/2} + O(t^{(k-1)/2}),\ \text{for } t \to \infty. \tag{5.14}$$

For the discussion of $A_{\nu,0}$ we observe that the system

$$g_n(x) = a_n \cos n_1 x_1 \dots \cos n_k x_k, \quad n \in \mathbb{N}_0^k, \tag{5.15}$$
$$a_n = \tfrac{1}{\| \cos n_1 x_1 \cdots \cos n_k x_k \|} = (\tfrac{2}{\pi})^{\frac{j}{2}} (\tfrac{1}{\pi})^{\frac{k-j}{2}},\ j = \#\{\text{nonzero entries in } n\},$$

is a complete orthonormal system in $L_2(Q_0)$. (This can be deduced from the completeness of $\{e^{in\cdot x}\}_{n\in\mathbb{Z}^k}$ by identifying $L_2(Q_0)$ with the subspace of $L_2(\mathbb{T}^k)$ consisting of functions that are *even* in each variable $x_1, \dots, x_k$.) Now the g_n are in fact eigenfunctions of $A_{\nu,0}$, with eigenvalues

$$\|n\|^2 = n_1^2 + \cdots + n_k^2, \quad n \in \mathbb{N}_0^k. \tag{5.16}$$

Again the completeness of the system (5.15) assures that we have found all eigenvalues and all eigenfunctions (up to linear combinations).

Also here one finds (5.14) by geometric considerations on grid points. In particular, it may be observed that

$$N(t; A_{\gamma,0}) \le 2^{-k}\omega_k t^{k/2} \le N(t; A_{\nu,0}) \text{ for all } t, \tag{5.17}$$

since the former equals the number of grid points in $(\mathbb{R}_+)^k \cap \overline{B}(0, t^{\frac{1}{2}})$, and the latter equals the number of grid points in $(\overline{\mathbb{R}}_+)^k \cap \overline{B}(0, t^{\frac{1}{2}})$.

In order to study the relation of these operators to the two variational operators A_γ and A_ν defined from the sesquilinear form

$$s(u, v) = \sum_{j=1}^{k} (D_j u, D_j v)_{L_2(Q_0)} \tag{5.18}$$

on $V = H_0^1(Q_0)$ resp. $H^1(Q_0)$, with $H = L_2(Q_0)$, we need a discussion of Sobolev spaces and boundary values. We first observe:

Lemma 5.1. *For any $m \in \mathbb{N}_0$, $C^\infty(\overline{Q}_0)$ is dense in $H^m(Q_0)$.*

Proof. The proof goes almost as in the smooth case, thanks to the simple geometric form of Q_0. Let $u \in H^m(Q_0)$. Instead of translating, we dilate u (blow it up), approximating $u(x)$ by

$$u_h(x) = u(\tfrac{1}{1+h}(x - p_{\frac{1}{2}}) + p_{\frac{1}{2}}), \quad p_{\frac{1}{2}} = (\tfrac{\pi}{2}, \dots, \tfrac{\pi}{2}),\ h > 0;$$

this converges to u in H^m on Q_0 for $h \to 0$. Here u_h is defined on $Q_h =]-\frac{h\pi}{2}, \pi + \frac{h\pi}{2}[^k$. Now approximate $e_{Q_h} u_h$ by $h_j * (e_{Q_h} u_h)$ for $j \to \infty$; it is seen to converge in $H^m(Q_0)$ similarly as in Theorem 3.2, using that

$$\langle D^\alpha (h_j * (e_{Q_h} u_h)), \varphi \rangle_{Q_0} = \langle h_j * (e_{Q_h} D^\alpha u_h), \varphi \rangle_{Q_0},$$

for $|\alpha| \le m$, $\varphi \in C_0^\infty(Q_0)$. □

Such density statements can be shown for quite general sets, namely the sets Ω having the so-called segment property, cf. e.g. [A 1965] or Edmunds and Edwards [E-E 1987].

For each $m \in \mathbb{N}_0$, the space $H^m(Q_0)$ can, besides the identification with $r_{Q_0} H^m(\mathbb{R}^k)$, be regarded as the space of restrictions to Q_0 of functions in $H^m(\mathbb{T}^k)$. We can also define a continuous linear extension operator $p_{(m),\mathbb{T}}$ from $H^m(Q_0)$ to $H^m(\mathbb{T}^k)$, as follows:

In view of Lemma 5.1, it suffices to define $p_{(m),\mathbb{T}}$ on the functions in $C^\infty(\overline{Q}_0)$ and show that it defines a continuous mapping. For $u \in C^\infty(\overline{Q}_0)$, we use the "reflection" described in Theorem 3.3 in several consecutive steps: First reflect across the boundary surfaces where $x_1 = 0$ and $x_1 = \pi$, just for a small distance a into $x_1 < 0$ resp. $x_1 > \pi$, so that only the values of u for, say, $x_1 \in [0,1]$ resp. $x_1 \in [\pi - 1, \pi]$ are used. Take $a < \pi/2$. Multiplying with a cut-off function of x_1 that is 1 on a neighborhood of $[0, \pi]$ and vanishes outside $[-\frac{a}{2}, \pi + \frac{a}{2}]$, we obtain a function defined for $x_1 \in [-\frac{\pi}{2}, \frac{3\pi}{2}]$ that we can extend by periodicity in x_1 to a function on $\mathbb{R} \times [0, \pi]^{k-1}$, such that it is C^{m-1} with piecewise continuous m'th derivatives (having their possible jumps at $x_1 = l\pi$, $l \in \mathbb{Z}$). Next, extend this function across $x_2 = 0$ and $x_2 = \pi$ in a similar way, and continue the procedure successively in all the other directions $x_3, \dots, x_k$. This gives a function that is in H^m on each bounded set, with the distribution derivatives up to order m coinciding with the (piecewise continuous) usual dervatives (checked e.g. as in [MA, Lemma 6.6]). The continuity of the mapping with respect to m-norm can be followed in each step.

Observe that for $m = 1$, there is a still better procedure: We can extend u as an *even* 2π-periodic function in each direction; let us call the resulting function $\approx{u}$. (So $\approx{u}$ takes the same value at $(x_1, x_2, \dots, x_k)$, $(-x_1, x_2, \dots, x_k)$ and $(2\pi l x_1, x_2, \dots, x_k)$, for all $l \in \mathbb{Z}$; there is a similar symmetry in the other coordinates.) The mapping $u \mapsto \approx{u}$ is continuous from $H^1(Q_0)$ to $H^1(\mathbb{T}^k)$, since this continuity holds for $u \in C^1(\overline{Q}_0)$.

Now γ_j can for $u \in H^m(Q_0)$ be defined on each face by reference to the definition on $H^m(\mathbb{T}^k)$. For example, the action of γ_j on the face $\{0\} \times Q_0^{(k-1)}$,

$Q_0^{(k-1)} =]0,\pi[^{\,k-1}$, is defined as the composition of

$$u \mapsto p_{(m),\mathbb{T}}u, \text{ from } H^m(Q_0) \text{ to } H^m(\mathbb{T}^k),$$

$$p_{(m),\mathbb{T}}u \mapsto D^j_{x_1}p_{(m),\mathbb{T}}u|_{x_1=0}, \text{ from } H^m(\mathbb{T}^k) \text{ to } H^{m-j-\frac12}(\mathbb{T}^{k-1}),$$

$$D^j_{x_1}p_{(m),\mathbb{T}}u|_{x_1=0} \mapsto r_{Q_0^{(k-1)}}(D^j_{x_1}p_{(m),\mathbb{T}}u|_{x_1=0}),$$
$$\text{from } H^{m-j-\frac12}(\mathbb{T}^{k-1}) \text{ to } H^{m-j-\frac12}(Q_0^{(k-1)}).$$

Altogether,

$$\gamma_j \text{ at } x_1=0 \text{ maps } H^m(Q_0) \text{ continuously into } H^{m-j-\frac12}(Q_0^{(k-1)}). \quad (5.23)$$

The full mapping γ_j is *not* surjective onto the union of the spaces $H^{m-j-1/2}$ for each face; there are compatibility requirements at the edges and corners where the faces meet. We shall not attempt a detailed study here.

Concerning $H_0^m(Q_0)$, the closure of $C_0^\infty(Q_0)$ in $H^m(Q_0)$, we observe that since γ_j is well-defined on each face, as in (5.23), and gives 0 on $C_0^\infty(Q_0)$, it gives 0 on $H_0^m(Q_0)$, for $j \le m-1$. Moreover, the mapping $e_{Q_0,\mathbb{T}}$ that extends by 0 on $[-\pi/2, 3\pi/2]^k \setminus Q_0$, and extends the resulting function to have period 2π in each coordinate, is continuous:

$$e_{Q_0,\mathbb{T}}\colon H_0^m(Q_0) \to H^m(\mathbb{T}^k).$$

For, it has this continuity when applied to $C_0^\infty(Q_0)$ and this extends by closure to all of $H_0^m(Q_0)$. We shall show:

Lemma 5.2. *The functions $u \in C^m(\overline{Q}_0) \cap H_0^m(Q_0)$ are precisely the functions in $C^m(\overline{Q}_0)$ that have $\gamma_j u = 0$ for $j \le m-1$ on all the boundary faces of Q_0.*

Proof. When $u \in C^m(\overline{Q}_0) \cap H_0^m(Q_0)$, then $\gamma_j u$ is zero on each boundary face, as noted before the lemma. To show the converse, let $u \in C^m(\overline{Q}_0)$ with $\gamma_j u = 0$ on each boundary face, $j \le m-1$. We have to show that this element of $H^m(Q_0)$ can be approximated there by $C_0^\infty(Q_0)$ functions. Now $e_{Q_0,\mathbb{T}}u$ identifies with a function in $H^m(\mathbb{T}^k)$, since the derivatives up to order $m-1$ are continuous and the m'th derivatives have nice jumps (as in the construction of $p_{(m),\mathbb{T}}$). By dilation (contraction) of u, defining

$$v_h(x) = u(\tfrac{1}{1-h}(x-p_{\frac12}) + p_{\frac12}), \quad p_{\frac12} = (\tfrac{\pi}{2},\dots,\tfrac{\pi}{2}), \quad h \in]0,1[\,,$$

and extension by zero on $[-\frac{\pi}{2}, \frac{3\pi}{2}]^k \setminus Q'_h$, $Q'_h =]\frac{h\pi}{2}, \pi - \frac{h\pi}{2}[$, extending further by periodicity, we get a sequence of functions converging in $H^m(\mathbb{T}^k)$ to the extension of u. Since v_h on $[-\frac{\pi}{2}, \frac{3\pi}{2}]^k$ is supported in the compact subset $\overline{Q'}_h$ of Q_0, convolution by h_j gives an approximating sequence of $C_0^\infty(Q_0)$ functions for sufficiently large j. □

This suffices to motivate that $H_0^m(Q_0)$ represents the boundary condition $\gamma_0 u = \gamma_1 u = \dots = \gamma_{m-1} u = 0$, although we have not fully analyzed the most general elements.

We shall now show:

Theorem 5.3.

1° The closure of $A_{\gamma,0}$ equals A_γ, the selfadjoint operator defined by the variational construction with s as in (5.18)*, $V = H_0^1(Q_0)$, $H = L_2(Q_0)$.*

2° The closure of $A_{\nu,0}$ equals A_ν, the selfadjoint operator defined by the variational construction with s as in (5.18)*, $V = H^1(Q_0)$, $H = L_2(Q_0)$.*

3° The system $\{f_n\}_{n\in\mathbb{N}^k}$ in (5.12) *is the full system of eigenvectors for A_γ, with eigenvalues $\|n\|^2$. Thus in the Fourier representation $F_\gamma \colon L_2(Q_0) \to \ell_2(\mathbb{N}^k)$ determined by the orthonormal basis $\{f_n\}_{n\in\mathbb{N}^k}$, A_γ corresponds to multiplication by $\|n\|^2$:*

$$A_\gamma = F_\gamma^{-1} M_{\|n\|^2} F_\gamma. \tag{5.24}$$

The system $\{g_n\}_{n\in\mathbb{N}_0^k}$ in (5.15) *is the full system of eigenvectors for A_ν, with eigenvalues $\|n\|^2$. Thus in the Fourier representation $F_\nu \colon L_2(Q_0) \to \ell_2(\mathbb{N}_0^k)$ determined by the orthonormal basis $\{g_n\}_{n\in\mathbb{N}_0^k}$,*

$$A_\nu = F_\nu^{-1} M_{\|n\|^2} F_\nu. \tag{5.25}$$

4° The domains $D(A_\gamma)$ and $D(A_\nu)$ are contained in $H^2(Q_0)$.

Proof. 1°. We denote $-\Delta = A$. Since $A_{\gamma,0} \subset A_{\max}$, $A_{\gamma,0}$ is closable. Recall from Section 1.3 that A_γ acts like $A_{\max}$ with $D(A_\gamma) = H_0^1(Q_0) \cap D(A_{\max})$; then since clearly $D(A_{\gamma,0}) \subset H_0^1(Q_0) \cap D(A_{\max})$, $A_{\gamma,0} \subset A_\gamma$. Since A_γ is closed,

$$\overline{A}_{\gamma,0} \subset A_\gamma. \tag{5.26}$$

Since (5.26) implies $A_\gamma \subset (\overline{A}_{\gamma,0})^*$, $\overline{A}_{\gamma,0} = A_\gamma$ will follow if we show that $\overline{A}_{\gamma,0}$ is selfadjoint. This is done by using the eigenfunction information we already have. Indeed, we shall show that (with $\lambda_n = \|n\|^2$, the eigenvalue at f_n)

$$\begin{gathered} u \in D(\overline{A}_{\gamma,0}) \iff \sum_{n\in\mathbb{N}^k} \lambda_n^2 |(u, f_n)|^2 < \infty, \\ \overline{A}_{\gamma,0} u = \sum_{n\in\mathbb{N}^k} \lambda_n (u, f_n) f_n \text{ in the affirmative case.} \end{gathered} \tag{5.27}$$

When (5.27) holds, it shows that $\overline{A}_{\gamma,0}$ corresponds to the multiplication operator M_{λ_n} on $\ell_2(\mathbb{N}^k)$ in the Fourier representation $F_\gamma : L_2(Q_0) \overset{\sim}{\to} \ell_2(\mathbb{N}^k)$

sending v into its Fourier coefficients $\{(v,f_n)\}_{n\in\mathbb{N}^k}$. The operator M_{λ_n} is selfadjoint, so since F_γ is unitary, it follows that $\overline{A}_{\gamma,0}$ is selfadjoint too.

It remains to show (5.27). When u is such that $\sum_{n\in\mathbb{N}^k}\lambda_n^2|(u,f_n)|^2<\infty$, then $u_N=\sum_{\|n\|\le N}(u,f_n)f_n\in D(A_{\gamma,0})$ converges to u for $N\to\infty$, and $A_{\gamma,0}u_N=\sum_{\|n\|\le N}\lambda_n(u,f_n)f_n$ converges in L_2 to a function $g=\sum_{n\in\mathbb{N}^k}\lambda_n(u,f_n)f_n$, so $u\in D(\overline{A}_{\gamma,0})$ with $\overline{A}_{\gamma,0}u=g$. Conversely, when $u\in D(\overline{A}_{\gamma,0})$, there is a sequence $v_N\to u$ such that $v_N\in D(A_{\gamma,0})$ and $A_{\gamma,0}v_N\to\overline{A}_{\gamma,0}u$. Then for any $n\in\mathbb{N}^k$,

$$\begin{aligned}(\overline{A}_{\gamma,0}u,f_n)&=\lim_N(A_{\gamma,0}v_N,f_n)=\lim_N(v_N,A_{\gamma,0}f_n)\\&=\lambda_n\lim_N(v_N,f_n)=\lambda_n(u,f_n),\end{aligned}$$

so $\overline{A}_{\gamma,0}u$ has the Fourier expansion $\overline{A}_{\gamma,0}u=\sum_{n\in\mathbb{N}^k}\lambda_n(u,f_n)f_n$, and $\|\overline{A}_{\gamma,0}u\|^2=\sum_{n\in\mathbb{N}^k}\lambda_n^2|(u,f_n)|^2$ (the Parseval equation). This ends the proof of 1°.

For 2°, we find in exactly the same way that $\overline{A}_{\nu,0}$ is selfadjoint, and equals $F_\nu^{-1}M_{\|n\|^2}F_\nu$. To see that $\overline{A}_{\nu,0}\subset A_\nu$, which will end the proof of 2°, we proceed as follows:

Clearly, $D(A_{\nu,0})\subset H^1(Q_0)$. When $u\in D(A_{\nu,0})$ and $v\in C^1(\overline{Q}_0)$, an integration by part shows that

$$(A_{\nu,0}u,v)=s(u,v). \tag{5.28}$$

Since $C^1(\overline{Q}_0)$ is dense in $H^1(Q_0)$ by Lemma 5.1, (5.28) extends to be valid for all $v\in H^1(Q_0)$. Then $u\in D(A_\nu)$, by definition. So $D(A_{\nu,0})\subset D(A_\nu)$, and since $A_{\nu,0}$ and A_ν both act like $A_{\max}$, $A_{\nu,0}\subset A_\nu$. Since the latter is closed, $\overline{A}_{\nu,0}\subset A_\nu$. This ends the proof of 2°.

In the course of these proofs, we have also obtained 3°.

Finally, 4° is deduced for A_γ from (5.27) as follows: A function u satisfying $\sum_{n\in\mathbb{N}^k}\lambda_n^2|(u,f_n)|^2<\infty$ is the restrictions to Q_0 of an odd periodic function $\widetilde{u}$ with an expansion $\widetilde{u}=\sum_{n\in\mathbb{Z}^k}c_ne_n$ in the full trigonometric system, such that $\sum_{n\in\mathbb{Z}^k}\|n\|^4|c_n|^2<\infty$. Hence $\widetilde{u}\in H^2(\mathbb{T}^k)$, and therefore $u\in H^2(Q_0)$. The proof for A_ν is similar, except that it refers to even periodic extensions. □

Note in particular that $N(t;A_\gamma)=N(t;A_{\gamma,0})$ and $N(t;A_\nu)=N(t;A_{\nu,0})$, so (5.14) and (5.17) hold for these counting functions also.

We have here shown some properties of A_γ and A_ν by "hand calculation" (using the special symmetries of the considered case), that one can show in general with a much greater effort, which we shall now take up.

5.4 Compact selfadjoint operators.

A compact operator in a Hilbert spaces H is an operator $T \in \mathbf{B}(H)$ that maps bounded sets into precompact sets. In other words, when x_k ($k \in \mathbb{N}$) is a bounded sequence, then Tx_k has a convergent subsequence. A compact normal operator can be diagonalized by a complete system of eigenvectors, such that the corresponding eigenvalues go to 0 at ∞, see e.g. G. K. Pedersen [P 1989, 3.3.5 and 3.3.8]. Let us here give a selfcontained proof that also shows how the eigenvalues are found (in principle), in the case of operators that are ≥ 0. (The general selfadjoint case can be treated very similarly, see e.g. Edmunds and Evans [E-E 1988, Th. II 5.2].) We first show:

Lemma 5.4. *Let $T \in \mathbf{B}(H)$ be selfadjoint ≥ 0. Then*

$$\|T\| = \sup\{\, (Tx, x) \mid \|x\| \leq 1 \,\}. \tag{5.30}$$

Proof. Denote the right hand side of (5.30) by μ. The inequality $\|T\| \geq \mu$ follows since

$$(Tx, x) \leq \|Tx\|\,\|x\| \leq \|T\|\,\|x\|^2 \leq \|T\| \text{ for } \|x\| \leq 1.$$

To show the other inequality we use the identity

$$4\|Tx\|^2 = (T(ax + \tfrac{1}{a}Tx), ax + \tfrac{1}{a}Tx) - (T(ax - \tfrac{1}{a}Tx), ax - \tfrac{1}{a}Tx), \tag{5.31}$$

that is valid for all $a > 0$, $x \in H$. It implies

$$4\|Tx\|^2 \leq \mu(\|ax + \tfrac{1}{a}Tx\|^2 + \|ax - \tfrac{1}{a}Tx\|^2) \leq 2\mu(a^2\|x\|^2 + \tfrac{1}{a^2}\|Tx\|^2).$$

When $Tx \neq 0$, the expression $a^2\|x\|^2 + \frac{1}{a^2}\|Tx\|^2$ takes its minimum for $a^2 = \|Tx\|/\|x\|$, and this choice gives that

$$4\|Tx\|^2 \leq 4\mu\|x\|\,\|Tx\|,$$

and hence

$$\|Tx\| \leq \mu\|x\|.$$

This obviously also holds if $\|Tx\| = 0$, and the validity for all x implies that $\|T\| \leq \mu$. $\square$

Theorem 5.5. *Let H be a Hilbert space, and let $T \in \mathbf{B}(H)$ be compact, selfadjoint and ≥ 0. Then T is diagonalizable with respect to an orthonormal basis $\{e_j\}_{j\in J}$ of H, such that $J = J_0 \cup J_1$, where $\{e_j\}_{j\in J_0}$ is a basis of the zero eigenspace and J_1 is countable (finite or infinite) with $\{e_j\}_{j\in J_1}$ consisting of eigenvectors belonging to positive eigenvalues of finite multiplicity. Here J_1 can be ordered (replaced by the infinite sequence $\mathbb{N}$ or by a set $\{1,\dots,N\}$) such that the corresponding eigenvalues μ_j form a decreasing sequence, going to 0 if the sequence is infinite:*

$$\mu_1 \geq \mu_2 \geq \cdots \geq \mu_j \geq \cdots \to 0, \quad \mu_j > 0. \tag{5.33}$$

When the eigenvalues and eigenvectors are ordered in this way, the n'th eigenvalue satisfies

$$\begin{aligned} \mu_n &= \max\{\, (Tx,x) \mid \|x\| \leq 1,\ x \perp e_1, \dots e_{n-1} \,\} \\ &= \min_{\substack{X \subset H \\ \dim X \leq n-1}} \ \max_{\substack{x \in H\setminus\{0\} \\ x \perp X}} \frac{(Tx,x)}{\|x\|^2}. \end{aligned} \tag{5.34}$$

Proof. (The proof has much in common with the proof in Mat 2MA V §2.2 showing the existence of eigenvalues and eigenvectors of a solution operator in the Sturm-Liouville theory.)

We shall first show that

$$\mu_1 = \sup\{\, (Tx,x) \mid \|x\| \leq 1 \,\}$$

is an eigenvalue. This surely holds if $T = 0$, so we just have to consider the case $T \neq 0$; then $\mu_1 > 0$ in view of Lemma 5.4. By definition, there is a sequence y_k with $\|y_k\| \leq 1$ such that $(Ty_k, y_k) \to \mu_1$. Then $(Ty_k, y_k) \neq 0$ from a certain step on, and hence also $y_k \neq 0$, so $x_k = y_k/\|y_k\|$ satisfies

$$(Ty_k, y_k) = (Tx_k, x_k)\|y_k\|^2 \leq (Tx_k, x_k) \leq \mu_1;$$

and therefore

$$(Tx_k, x_k) \to \mu_1 \text{ for } k \to \infty; \quad \|x_k\| = 1.$$

By the compactness of T, Tx_k has a convergent subsequence $Tx_{k_j} \to v$ in H. Now

$$\begin{aligned} 0 \leq \|Tx_{k_j} - \mu_1 x_{k_j}\|^2 &= \|Tx_{k_j}\|^2 - 2\mu_1 (Tx_{k_j}, x_{k_j}) + \mu_1^2 \\ &\leq 2\mu_1^2 - 2\mu_1 (Tx_{k_j}, x_{k_j}) \to 0, \end{aligned}$$

so in fact $\|Tx_{k_j} - \mu_1 x_{k_j}\| \to 0$, and

$$\mu_1 x_{k_j} = Tx_{k_j} - (Tx_{k_j} - \mu_1 x_{k_j}) \to v.$$

Since $x_{k_j} \to \frac{1}{\mu_1} v$ and $Tx_{k_j} \to v$, v is an eigenvector for T with the eigenvalue μ_1. Since $\|x_{k_j}\| = 1$, $\|v\| = \mu_1$; we shall denote $v/\|v\| = e_1$.

The eigenvalue μ_1 is the largest possible, since any normed eigenvector e with eigenvalue μ must satisfy

$$\mu_1 \geq (Te, e) = (\mu e, e) = \mu. \tag{5.35}$$

Insertion shows that μ_1 satisfies (5.34) (with the empty set of eigenvectors with lower index, and with $X = \{0\}$).

To find the next eigenvalue, we observe that the space X_1 spanned by e_1 is invariant under T (obviously), and so is its orthogonal complement $H_1 = X_1^\perp$:

$$(Tx, e_1) = (x, Te_1) = \mu_1 (x, e_1) = 0, \text{ when } x \in H_1.$$

Then the restriction T_1 of T to H_1 is a compact selfadjoint nonnegative operator in H_1.

To this we can apply the above procedure, finding that

$$\mu_2 = \sup\{\, (T_1 x, x) \mid \|x\| \leq 1,\ x \in H_1 \,\}$$

is an eigenvalue of T_1, hence of T, with normed eigenvector $e_2 \in H_1$. It certainly verifies the first line in (5.34); the second line will be dealt with at the end of the proof. Moreover, $\mu_2 \leq \mu_1$, in view of (5.35).

One repeats this procedure as long as T_n is nonzero. In the $n+1$'st step, we set $X_n = \operatorname{span}\{e_1, \dots, e_n\}$ and $H_n = X_n^\perp$; it is invariant under T since

$$\begin{aligned}(Tx, c_1 e_1 + \cdots + c_n e_n) &= (x, T(c_1 e_1 + \cdots + c_n e_n)) \\ &= (x, \mu_1 c_1 e_1 + \cdots + \mu_n c_n e_n) = 0 \text{ when } x \perp X_n.\end{aligned}$$

Thus T restricts to an operator T_n on H_n that is again compact selfadjoint ≥ 0, and to which the procedure applies. This gives the $n+1$'st eigenvalue μ_{n+1}, which is $\leq \mu_n$ and satisfies the first line in (5.34).

If there is a number n_0 such that $\mu_{n_0} > 0$ and T_{n_0} is the zero operator, then H_{n_0} equals the nullspace $Z(T)$ of T, i.e. the zero eigenspace. For, it is clear that $H_{n_0} \subset Z(T)$, and on the other hand, if $Tx = 0$, then for $n \leq n_0$,

$$(x, e_n) = \tfrac{1}{\mu_n}(x, Te_n) = \tfrac{1}{\mu_n}(Tx, e_n) = 0, \tag{5.36}$$

so $x \perp X_{n_0}$, hence lies in H_{n_0}. This is the case formulated in the theorem, where J_1 is finite with $N = n_0$; the $\{e_j\}_{j\in J_0}$ can be taken as an orthonormal basis of H_{n_0}.

Otherwise the procedure goes on to give an infinite, decreasing sequence of positive eigenvalues μ_n, with an orthonormal system of eigenfunctions e_n, $n \in \mathbb{N}$. The sequence μ_n converges to a limit $c \geq 0$. If c were positive, we would have a bounded sequence $x_n = \frac{1}{\mu_n} e_n$ with $Tx_n = e_n$ being an orthonormal sequence; this has no convergent subsequence, in contradiction to the compactness of T. Thus $c = 0$, and (5.33) is shown. In particular, μ_n can only take the same positive value finitely many times, so there is only a finite number of e_n's belonging to the same positive eigenvalue.

Let V_1 be the closure of the space spanned by the e_n, and let $V_0 = V_1^\perp$; observe that $V_0 = \bigcap_{n\in\mathbb{N}} H_n$. Then V_0 is invariant under T, and the restriction of T to V_0, T_{V_0}, satisfies

$$0 \leq (T_{V_0}x, x) = (Tx, x) \leq \mu_n \|x\|^2 \text{ for all } n \in \mathbb{N},\ x \in V_0,$$

so $(T_{V_0}x, x) = 0$ on V_0. It follows by Lemma 5.4 that $T_{V_0} = 0$, i.e., $V_0 \subset Z(T)$. The converse, $Z(T) \subset V_0$, is seen as above by (5.36).

If we let $\{e_j\}_{j\in J_0}$ be an orthonormal basis of V_0, we have that $\{e_n\}_{n\in\mathbb{N}} \cup \{e_j\}_{j\in J_0}$ is an orthonormal basis of H.

It remains to account for the second formulation in (5.34). First we show that when X is an arbitrary finite dimensional subspace of H, then the maximum of $(Tx, x)/\|x\|^2$ for $x \in X^\perp \setminus \{0\}$ is *attained:* Let $P_{X^\perp}$ denote the orthogonal projection onto $X^\perp$, and let $T' = P_{X^\perp}T$, it is a compact operator in $X^\perp$ satisfying

$$(T'x, x) = (P_{X^\perp}Tx, x) = (Tx, x) \geq 0 \text{ for } x \in X^\perp,$$

hence selfadjoint ≥ 0 there. An application of the first step in the above proof to T' shows that the maximum of $(T'x, x)/\|x\|^2$ on $X^\perp \setminus \{0\}$ is attained.

Now let us denote

$$a_n = \inf_{\substack{X\subset H \\ \dim X \leq n-1}} \max_{\substack{x\in H\setminus\{0\} \\ x \perp X}} \frac{(Tx, x)}{\|x\|^2}; \tag{5.37}$$

we have to show that $\mu_n = a_n$ for each n, and that the infimum is attained, so that it is a minimum.

Since, by the first line in (5.34),

$$\mu_n = \max_{\substack{x\in H\setminus\{0\} \\ x\perp X_{n-1}}} \frac{(Tx, x)}{\|x\|^2},$$

where X_{n-1} is a special case of the X occurring in (5.37), $\mu_n \ge a_n$.

To show that $\mu_n \le a_n$, we must show that for an arbitrary n'-dimensional space $X \subset H$ with $n' \le n-1$,

$$\mu_n \le \max_{\substack{x \in H\setminus\{0\} \\ x \perp X}} \frac{(Tx,x)}{\|x\|^2}. \tag{5.38}$$

Let $x_1, \dots, x_{n'}$ be a basis of X. To find a vector $v = c_1e_1 + \dots + c_ne_n$ orthogonal to X, we have to solve the n' homogeneous equations with n unknowns $c_1, \dots, c_n$:

$$c_1(e_1, x_j) + \dots + c_n(e_n, x_j) = 0, \text{ for } j = 1, \dots, n';$$

since $n' < n$, this always has a nontrivial solution. Such a v satisfies

$$\begin{aligned}(Tv, v) = (\sum_{j\le n} Tc_je_j, \sum_{k\le n} c_ke_k) = \sum_{j\le n} \mu_j |c_j|^2 \\ \ge \mu_n \sum_{j\le n} |c_j|^2 = \mu_n \|v\|^2, \quad v \ne 0.\end{aligned}$$

This shows (5.38), and hence $a_n \ge \mu_n$. The infimum is attained at $X = X_{n-1}$. □

The second line in (5.34) is useful for comparison of eigenvalues of different operators. We have for example:

Corollary 5.6. *Let S and T be operators in $\mathbf{B}(H)$ that are compact, selfadjoint ≥ 0, and satify $S \le T$, i.e.,*

$$(Sx, x) \le (Tx, x) \text{ for all } x \in H. \tag{5.39}$$

Then the number ($\le \infty$) of nonzero eigenvalues $\mu_n(S)$ for S is $\le$ the number of nonzero eigenvalues $\mu_n(T)$ for T; and when they are ordered as in the above theorem, each nonzero eigenvalue of S satisfies

$$\mu_n(S) \le \mu_n(T). \tag{5.40}$$

Proof. This follows immediately from (5.39) and (5.34). □

There is also a converse of Theorem 5.5 (cf. [P 1989, Lemma 3.3.5]), for which we shall include the classical proof:

Theorem 5.7. *Let the Hilbert space H have an orthonormal basis $\{e_j\}_{j\in J}$, where $J = J_0 \cup J_1$, with J_1 finite or equal to $\mathbb{N}$, and let T be an operator acting as follows:*

$$Tx = \sum_{j\in J_1} \mu_j (x, e_j) e_j ,$$

where $\mu_j \neq 0$, and $\mu_j \to 0$ for $j \to \infty$ if J_1 is infinite. Then T is compact.

Proof. In the expression defining Tx, one could include a zero term $\sum_{j\in J_0} 0(x, e_j)e_j$. By the Parseval equation,

$$\|Tx\|^2 = \sum_{j\in J_1} |\mu_j (x, e_j)|^2 \leq \max |\mu_j|^2 \|x\|^2 ,$$

with equality when x is a vector e_j with largest $|\mu_j|$, so T is a bounded operator with $\|T\| = \max_j |\mu_j|$.

Let x_k be a sequence in H with $\|x_k\| \leq C$ for all $k \in \mathbb{N}$; we must show that Tx_k has a convergent subsequence. For this, note that when we write

$$Tx_k = \sum_{j\in J_1} \mu_j c_{jk} e_j , \quad c_{jk} = (x_k, e_j),$$

then $|c_{jk}| \leq C$ for all j, k, so each sequence $\{c_{jk}\}_{k\in\mathbb{N}}$ has a convergent subsequence. We use this as follows: Let

$$I_n = \{ j \in J_1 \mid |\mu_j| \geq \tfrac{1}{n} \}, \quad n \in \mathbb{N};$$

then $I_n \subset I_{n+1}$ for all n and $J_1 = \bigcup_{n\in\mathbb{N}} I_n$. Let

$$T_n x = \sum_{j\in I_n} \mu_j (x, e_j) e_j ,$$

then

$$\|Tx - T_n x\| = \| \sum_{j\in J_1 \setminus I_n} \mu_j (x, e_j) e_j \| \leq \tfrac{1}{n} \|x\|.$$

For $n = 1$, we can find a subsequence $\{k_l^1\}_{l\in\mathbb{N}}$ of $\{k\}_{k\in\mathbb{N}}$ such that the sequence of vectors $\{c_{jk_l^1}\}_{j\in I_1}$ converges to a limit $\{c_j\}_{j\in I_1}$ for $l \to \infty$; i.e., each sequence $c_{jk_l^1}$ converges to a limit c_j, for $j \in I_1$. Next, we construct a subsequence k_l^2 of $\{k_l^1\}_{l\in\mathbb{N}}$ such that also the sequences $c_{jk_l^2}$ with $j \in I_2 \setminus I_1$ converge to limits c_j for $l \to \infty$; then we have convergence for all $j \in I_2$. One goes on in this way; in the general step one takes for k_l^{n+1} a subsequence of k_l^n such that the sequences $c_{j,k_l^{n+1}}$ converge to limits c_j for $l \to \infty$, when

$j \in I_{n+1}$. If J_1 is finite, the procedure ends after a finite number of steps, and the last subsequence is index sequence for a convergent subsequence of $\{Tx_k\}$. If J_1 is infinite, we define from the sequences $\{k_l^n\}$, $n, l \ge 1$, the sequence $\{k_r^r\}_{r\in\mathbb{N}}$; it is called the *diagonal sequence.* For any of the sequences $\{k_r^n\}_{r\in\mathbb{N}}$, the sequence $\{k_r^r\}_{r\in\mathbb{N}}$ is a subsequence from $r = n$ and onwards, so $c_{j,k_r^r} \to c_j$ for $r \to \infty$, *for all* $j \in J_1$. Now we claim that $Tx_{k_r^r}$ is a Cauchy sequence in H, hence convergent. For,

$$\begin{aligned}\|Tx_{k_r^r} - Tx_{k_s^s}\| &\le \|(T - T_n)x_{k_r^r}\| + \|T_n(x_{k_r^r} - x_{k_s^s})\| + \|(T - T_n)x_{k_s^s}\| \\ &\le 2\tfrac{1}{n}C + \|T_n(x_{k_r^r} - x_{k_s^s})\|.\end{aligned}$$

For a given ε we first choose n so large that $2C/n \le \varepsilon/2$; then since I_n is finite and each coefficient sequences c_{j,k_r^r} converges for $r \to \infty$, there exists N so that $\|T_n(x_{k_r^r} - x_{k_s^s})\| \le \varepsilon/2$ for $r, s \ge N$. $\square$

The method of successively taking subsequences can also be used to show the general fact that if a sequence of finite rank operators converges in operator norm to an operator T, then T is compact.

5.5 Applications to variational operators.

We shall now apply the above results to selfadjoint unbounded operators. The interesting case is where the Hilbert space H is infinite dimensional (so that the operators *can* be unbounded), we assume this from now on.

As recalled in Chapter 1, a triple (H, V, s) as in Theorem 1.4 with a *symmetric* coercive sesquilinear form gives rise to a *selfadjoint* operator S by the variational construction. Conversely, any selfadjoint operator S in H with lower bound $m(S) > -\infty$ is the variational operator defined from a suitable triple, namely the one where V is the completion of $D(S)$ with respect to the scalar product

$$(u, v)_V = (Su, v)_H + (c - m(S))(u, v)_H, \quad u, v \in D(S), \quad c > 0; \tag{5.40}$$

and $s(u, v)$ is the continuous extension of the sesquilinear form $(Su, v)_H$ to all of V, cf. e.g. [MA, Sect. 2.6] (the analysis in the proof of the Friedrichs theorem assures that V identifies with a subspace of H). Any $c > 0$ can be used; they all give equivalent scalar products on V. V is often called the *form domain* for S.

Assume for simplicity that $m(S) \ge 1$, and take $c = m(S)$ in (5.40). Then S^{-1} exists and is a selfadjoint bounded operator ≥ 0; and $(u, v)_V = s(u, v)$.

Now consider the case where the injection $V \hookrightarrow H$ is *compact*, i.e., any bounded sequence in V has a subsequence that is convergent in H. This implies that S^{-1} *is compact.* For, if x_k ($k \in \mathbb{N}$) is a sequence in H such that

$\|x_k\|_H \le C$ for all k, then $y_k = S^{-1}x_k$ lies in $D(S) \subset V$ and satisfies

$$\begin{aligned} s(y_k, y_k) &= (Sy_k, y_k)_H = (x_k, S^{-1}x_k)_H \\ &\le \|x_k\|_H \, \|S^{-1}x_k\|_H \le C^2\|S^{-1}\|; \end{aligned}$$

hence y_k is bounded in V-norm, and then, by the compactness of the injection, has a subsequence that converges in H. We shall see later that the converse also holds: compactness of S^{-1} implies compactness of $V \hookrightarrow H$.

When S^{-1} is compact, we can apply the results of Section 5.4. Since S^{-1} has nullspace $\{0\}$, it is diagonalized by a complete orthonormal system of eigenvectors $\{e_n\}_{n\in\mathbb{N}}$, with eigenvalues μ_n forming a decreasing sequence as in (5.33). (Recall that we have taken H to be infinite dimensional; and note that we now find that it must be separable.)

So S^{-1} acts as follows:

$$S^{-1}x = \sum_{n\in\mathbb{N}} \mu_n(x, e_n)_H e_n, \text{ for } x \in H;$$

it corresponds to the multiplication operator M_{μ_n} in the Fourier representation $F\colon H \to \ell_2(\mathbb{N})$ that maps x into its Fourier coefficients $\{(x, e_n)_H\}_{n\in\mathbb{N}}$.

It follows immediately that S corresponds to the multiplication operator M_{λ_n}, where

$$\lambda_n = \tfrac{1}{\mu_n}, \quad 0 < \lambda_1 \le \lambda_2 \le \dots \le \lambda_n \le \dots \to \infty; \tag{5.41}$$

here

$$\begin{aligned} D(S) &= \{u \in H \mid \sum_{n\in\mathbb{N}} \lambda_n^2 |(u, e_n)_H|^2 < \infty\}, \\ Su &= \sum_{n\in\mathbb{N}} \lambda_n (u, e_n)_H e_n. \end{aligned} \tag{5.42}$$

We have shown:

Proposition 5.8. *Let S be selfadjoint ≥ 1 in H, and let $s(u, v)$ be the associated sesquilinear form with domain $V \subset H$, such that S is the variational operator determined by (H, V, s).*

If the injection $V \hookrightarrow H$ is compact, then S^{-1} is a compact operator in H.

When S^{-1} is compact, S has a complete orthonormal system of eigenvectors (the same as those of S^{-1}) with eigenvalues (5.41)*, and diagonalizes as in* (5.42).

The sesquilinear form $s(u, v)$ and the space V are related to the orthogonal expansions by:

Proposition 5.9. *Let S, s and V be as in Proposition 5.8, with S^{-1} compact.*

The space V consists of precisely the elements v in H for which

$$\sum_{n\in\mathbb{N}} \lambda_n |(v,e_n)_H|^2 < \infty; \tag{5.44}$$

for these elements, the ordinary Fourier series $\sum_{n\in\mathbb{N}}(v,e_n)_H e_n$ converges to v in V. Moreover,

$$s(u,v) = \sum_{n\in\mathbb{N}} \lambda_n (u,e_n)_H (e_n,v)_H, \text{ when } u,v\in V. \tag{5.45}$$

Proof. Let us take $s(u,v)$ as scalar product on V.

We begin by observing that the functions $v_n = \frac{1}{\sqrt{\lambda_n}} e_n$ form an orthonormal system in V:

$$(v_n,v_m)_V = s(v_n,v_m) = (S\tfrac{1}{\sqrt{\lambda_n}}e_n, \tfrac{1}{\sqrt{\lambda_m}}e_m)_H = (e_n,e_m)_H = \delta_{nm}.$$

It is complete, since one has for $v\in V$:

$$(v,v_n)_V = s(v,v_n) = (v,Sv_n)_H = \sqrt{\lambda_n}(v,e_n)_H, \tag{5.46}$$

so if v is V-orthogonal to all v_n, it is H-orthogonal to all e_n and hence equals 0. The Parseval equation gives for each $v\in V$, using (5.46) again:

$$\|v\|_V^2 = \sum_{n\in\mathbb{N}} |s(v,v_n)|^2 = \sum_{n\in\mathbb{N}} \lambda_n |(v,e_n)_H|^2,$$

showing that for $v\in V$, (5.44) holds.

Conversely, if $v\in H$ is such that (5.44) holds, then

$$w_m = \textstyle\sum_{n\le m}(v,e_n)_H e_n$$

satisfies

$$\begin{aligned}\|w_{m+p}-w_m\|_V^2 &= s(\textstyle\sum_{n=m+1}^{m+p}(v,e_n)_H e_n, \sum_{j=m+1}^{m+p}(v,e_j)_H e_j)\\ &= \textstyle\sum_{n=m+1}^{m+p}\lambda_n|(v,e_n)_H|^2,\end{aligned}$$

again by (5.46), and hence w_m is a Cauchy sequence in V for $m\to\infty$. So it converges to some w in V and hence to w in H; and since it converges to v in H, v must equal w and lies in V.

This shows the first claim in the proposition. For the second claim, we now have that

$$\begin{aligned}s(u,v) &= \lim_{m\to\infty} s(\textstyle\sum_{n\le m}(u,e_n)_H e_n, \sum_{j\le m}(v,e_j)_H e_j)\\ &= \lim_{m\to\infty}\textstyle\sum_{n\le m}\lambda_n (u,e_n)_H(e_n,v)_H.\end{aligned}$$ □

This allows us to give a characterization of the eigenvalues of S in terms of the associated sesquilinear form:

Theorem 5.10. *When S is selfadjoint ≥ 1 in H with S^{-1} compact, and s is the associated sesquilinear form, with domain V, the eigenvalues λ_n of S (cf. (5.41) ff.) are described by*

$$\begin{aligned}\lambda_n &= \min_{\substack{v\in V\setminus\{0\}\\ v\perp e_1,\dots,e_{n-1}}} \frac{s(v,v)}{\|v\|_H^2}\\ &= \max_{\substack{X\subset H\\ \dim X\leq n-1}} \min_{\substack{v\in V\setminus\{0\}\\ v\perp X}} \frac{s(v,v)}{\|v\|_H^2}.\end{aligned} \tag{5.47}$$

The formula extends to the case where S is just selfadjoint lower bounded, associated with a coercive symmetric sesquilinear form $s(u,v)$ on a space $V\subset H$ as in Theorem 1.4, as long as the injection of V into H is compact.

Proof. The first line in (5.47) follows easily from Proposition 5.9: When $v\in V$ with $v\perp e_1,\dots e_{n-1}$, then

$$s(v,v)=\sum_{j\geq n}\lambda_j|(v,e_j)_H|^2\geq \lambda_n\sum_{j\geq n}|(v,e_j)_H|^2=\lambda_n\|v\|_H^2,$$

since the sequence λ_n is increasing; hence

$$\lambda_n\leq \frac{s(v,v)}{\|v\|_H^2}\text{ when } v\perp e_1,\dots,e_{n-1}.$$

Equality holds for $v=e_n$ (cf. (5.46)).

For the second line, we proceed as at the end of the proof of Theorem 5.5. To see that the minimum is attained, let $V'=V\cap X^\perp$, let H' be the closure of V' in H and let s' be the restriction of s to V'. This triple (H',V',s') has similar properties as the given triple (H,V,s), and the considered minimum equals $\min\{\, s'(v,v)/\|v\|_H^2 \mid v\in V'\setminus\{0\}\,\}$, so it is attained in view of the first part of the theorem. Denote

$$\sup_{\substack{X\subset H\\ \dim X\leq n-1}} \min_{\substack{v\in V\setminus\{0\}\\ v\perp X}} \frac{s(v,v)}{\|v\|_H^2}=b_n.$$

It is clear that $\lambda_n\leq b_n$, since the space spanned by $e_1,\dots,e_{n-1}$ is a special case of X. On the other hand, when a general X is considered, there is a nontrivial vector $w=c_1e_1+\dots+c_ne_n$ orthogonal to X, and evaluation of $s(w,w)/\|w\|_H^2$ gives a number $\leq\lambda_n$. Hence

$$\min_{\substack{v\in V\setminus\{0\}\\ v\perp X}} \frac{s(v,v)}{\|v\|_H^2}\leq\lambda_n$$

for all X, and thus $b_n \le \mu_n$. This shows that $\lambda_n = b_n$. The supremum is attained at $X = \operatorname{span}\{e_1, \dots, e_{n-1}\}$, hence is a maximum.

For the last statement in the theorem, we observe that the addition of a constant c times $(u,v)_H$ to $s(u,v)$ corresponds to the replacement of S by $S + c$, whereby the eigenvalues are shifted to $\lambda_n + c$. At the same time,

$$\frac{s(v,v) + c \cdot (v,v)_H}{\|v\|_H^2} = \frac{s(v,v)}{\|v\|_H^2} + c.$$

This allows us to reduce to the case where $s(u,v)$ is V-elliptic, as already treated. □

The expressions in the right hand side of (5.47) are often called the Rayleigh (or Rayleigh-Ritz) coefficients, and the whole statement is called the max-min principle. It helps us compare eigenvalues, when the operators stem from different sesquilinear forms on the same space V, and even different forms on different spaces V, when there is a suitable ordering.

Theorem 5.11. *Let (H_1, V_1, s_1) and (H_2, V_2, s_2) be triples giving rise to selfadjoint variational operators S_1 resp. S_2 as in Theorem 1.4. Assume that $V_1 \subset V_2$ with continuous injection, that H_1 is a closed subspace of H_2, that the injections of V_i into H_i are compact ($i = 1, 2$), and that $s_1(v,v) \ge s_2(v,v)$ for $v \in V_1$. Then the eigenvalues $\lambda_n(S_1)$ and $\lambda_n(S_2)$ of S_1 and S_2 (ordered as above) satisfy*

$$\lambda_n(S_1) \ge \lambda_n(S_2), \text{ for all } n \in \mathbb{N}. \tag{5.50}$$

Equivalently, the counting functions $N(t; S_i) = \#\{n \mid \lambda_n(S_i) \le t\}$ satisfy

$$N(t; S_1) \le N(t; S_2), \text{ for all } t \ge 0.$$

Proof. Theorem 5.10 applies to both S_1 and S_2. Note that when X_2 is a finite dimensional subspace of H_2, and $v \in H_1$, then $v \perp X_2 \iff v \perp X_1$, where $X_1 = P_{H_1} X_2$, orthogonal projection. All subspaces of H_1 of dimension $\le n-1$ are obtained as $P_{H_1} X$ when X runs through the subspaces of H_2 of dimension $\le n-1$.

For each $X \subset H_2$ of dimension $n-1$,

$$\begin{aligned}
\min\{\, s_1(v,v)/\|v\|_{H_1}^2 \mid v \in V_1 \setminus \{0\},\ v \perp P_{H_1} X \,\}& \\
\ge \min\{\, s_2(v,v)/\|v\|_{H_2}^2 \mid v \in V_1 \setminus \{0\},\ v \perp X \,\}& \\
\ge \min\{\, s_2(v,v)/\|v\|_{H_2}^2 \mid v \in V_2 \setminus \{0\},\ v \perp X \,\}&,
\end{aligned}$$

since $s_1(v,v) \ge s_2(v,v)$ on V_1, and V_2 contains more elements than V_1. Taking the maximum over all subspaces X of H_2 of dimension $\le n-1$, we get the n'th eigenvalues, which then must satisfy the inequality (5.50). □

Whereas the domains of two different unbounded selfadjoint operators rarely admit any inclusions, the form domains (V_1 and V_2) may very well have inclusions, with comparable sesquilinear forms. This is the case for example for the Dirichlet and Neumann realizations A_γ and A_ν of $-\Delta$ on a set Ω, where the form domains are $H_0^1(\Omega)$ resp. $H^1(\Omega)$, and the sesquilinear forms have the same expression $\sum_j (D_j u, D_j v)_0$. Provided that we have the required compact injection (see Section 5.6), Theorem 5.11 will give:

$$\lambda_n(A_\gamma) \geq \lambda_n(A_\nu), \text{ for all } n \in \mathbb{N}. \tag{5.51}$$

With the description of V given in Proposition 5.9, we can also show that S^{-1} compact implies $V \hookrightarrow H$ compact. In fact, Proposition 5.9 shows that in the situation there, V equals the domain of the operator $S^{\frac{1}{2}} \equiv F^{-1} M_{\lambda_n^{\frac{1}{2}}} F$, using the Fourier representation $F\colon H \overset{\sim}{\to} \ell_2(\mathbb{N})$ mentioned above. Equivalently, V is the range of the operator $S^{-\frac{1}{2}} \equiv F^{-1} M_{\lambda_n^{-\frac{1}{2}}} F$. Since $\lambda_n^{-\frac{1}{2}} \to 0$ for $n \to \infty$, $S^{-\frac{1}{2}}$ is a compact operator by Theorem 5.7. When v_k is a sequence in V, then $f_k = S^{\frac{1}{2}} v_k$ is a sequence in H such that $v_k = S^{-\frac{1}{2}} f_k$, and

$$\|v_k\|_V^2 = s(v_k, v_k) = \sum_{n \in \mathbb{N}} \lambda_n |(v_k, e_n)|^2 = \|S^{\frac{1}{2}} v_k\|_H^2 = \|f_k\|_H^2.$$

Here $\|v_k\|_V$ bounded $\implies$ $\|f_k\|_H$ bounded $\implies$ $S^{-\frac{1}{2}} f_k$ has a subsequence converging in H $\implies$ v_k has a subsequence converging in H.

5.6 Dirichlet-Neumann bracketing.

We are now in a position to study the spectrum of realizations of selfadjoint strongly elliptic operators A on general domains. We do this by comparison with simpler cases, using the max-min property of eigenvalues shown in Theorem 5.10; notably the consequences described in the subsequent theorm.

In preparation for the study, let us first show the compactness of suitable imbeddings.

Theorem 5.12. *For $s > 0$, $t \in \mathbb{R}$, the injection of $H^{s+t}(\mathbb{T}^k)$ into $H^t(\mathbb{T}^k)$ is compact.*

Proof. First let $t = 0$. As noted in Section 5.2, $H^s(\mathbb{T}^k)$ equals the domain of the operator

$$\Xi_s = F^{-1} M_{\langle n \rangle^s} F,$$

and

$$\|u\|_{s,\wedge} = \|\Xi_s u\|_{L_2(\mathbb{T}^k)}.$$

The operator $\Xi_{-s} = F^{-1}M_{\langle n\rangle^{-s}}F$ is a bounded operator in $L_2(\mathbb{T}^k)$; it equals Ξ_s^{-1}. In details,

$$\Xi_{-s}u = \sum_{n\in\mathbb{Z}^k} \langle n\rangle^{-s}(u, e_n)e_n.$$

Theorem 5.7 shows that Ξ_{-s} is a compact operator in $L_2(\mathbb{T}^k)$. Now when u_j is a bounded sequence in $H^s(\mathbb{T}^k)$, we have that $u_j = \Xi_{-s}f_j$, where $\|f_j\|_0$ is bounded, so the compactness of Ξ_{-s} in $L_2(\mathbb{T}^k)$ implies that u_j has a convergent subsequence in $L_2(\mathbb{T}^k)$.

This shows the statement for $t = 0$. For $t \neq 0$ we combine what was just proved with the unitary maps $\Lambda_t \colon H^{s+t}(\mathbb{T}^k) \overset{\sim}{\to} H^s(\mathbb{T}^k)$ and $\Lambda_t \colon H^t(\mathbb{T}^k) \overset{\sim}{\to} H^0(\mathbb{T}^k) = L_2(\mathbb{T}^k)$. $\square$

We can use this to get results for suitable subsets of $\mathbb{R}^k$.

Theorem 5.13. *Let* $m \in \mathbb{N}$.

1° *When* Ω *is a smooth bounded open subset of* $\mathbb{R}^k$, *the injection of* $H^m(\Omega)$ *into* $L_2(\Omega)$ *is compact.*

2° *When* Q *is a box,* $Q = \,]a_1, b_1[\, \times \cdots \times \,]a_k, b_k[\,$, *the injection of* $H^m(Q)$ *into* $L_2(Q)$ *is compact.*

3° *For any bounded open subset* Ω *of* $\mathbb{R}^k$, *the injection of* $H_0^m(\Omega)$ *into* $L_2(\Omega)$ *is compact.*

4° *When* Ω *is a finite union* $\Omega = \bigcup_{j=1}^N \Omega_j$ *of smooth bounded open sets and boxes as in* 1° *and* 2°, *the injection of* $H^m(\Omega)$ *into* $L_2(\Omega)$ *is compact.*

Proof. 1°. By a linear coordinate change, we can obtain that $\Omega \subset Q_0 = \,]0, \pi[^k$. There is a continuous extension operator $p_{(m),\Omega,\mathbb{T}}$ from $H^m(\Omega)$ to $H^m(\mathbb{T}^k)$, defined from the usual extension operator by composition with multiplication by a function $\eta \in C_0^\infty(\,]-\frac{\pi}{2}, \frac{3\pi}{2}[^k)$ that is 1 on $\overline{\Omega}$, and extension of the resulting function on $[-\frac{\pi}{2}, \frac{3\pi}{2}]$ to a function with period 2π in each coordinate. When u_j is bounded in $H^m(\Omega)$, $p_{(m),\Omega,\mathbb{T}}u_j$ is bounded in $H^m(\mathbb{T}^k)$. It has a convergent subsequence $p_{(m),\Omega,\mathbb{T}}u_{j_r}$ in $L_2(\mathbb{T}^k)$ $(r \in \mathbb{N})$ by the preceding theorem; and then u_{j_r} is convergent in $L_2(\Omega)$.

2°. By a linear coordinate change, we can reduce to the case $Q = Q_0$. Here the extension operator $p_{(m),\mathbb{T}}$ defined after Lemma 5.1 can be used, and the proof completed in the same way as above.

3°. Let R be so large that $\overline{\Omega} \subset B(0, R)$. The extension by zero, e_Ω, followed by restriction to $B(0, R)$, $r_{B(0,R)}$, maps $H_0^m(\Omega)$ continuously into $H^m(B(0, R))$, since it is continuous on the dense subset $C_0^\infty(\Omega)$. Then if $\{u_l\}_{l\in\mathbb{N}}$ is bounded in $H_0^m(\Omega)$, $r_{B(0,R)}e_\Omega u_l$ is bounded in $H^m(B(0, R))$, and hence by 1° has a subsequence that converges in $L_2(B(0, R))$, and in $L_2(\Omega)$.

4°. If a sequence u_l, $l \in \mathbb{N}$, is bounded in $H^m(\Omega)$, we can find a subsequence that converges in $L_2(\Omega)$ by successively, for $j = 1, \dots, N$, taking subsequences u_{l_r} of u_l such that $r_{\Omega_j} u_{l_r}$ converges in $L_2(\Omega_j)$; here we use 1° and 2°. □

Also the imbeddings $H^{s+t}(\Omega) \hookrightarrow H^t(\Omega)$ with $s > 0$, $t \in \mathbb{R}$, can be shown to be compact, when Ω is bounded and smooth, or is a cube. We also note that when only a fixed m is considered, one can make do with less smoothness of the domain (e.g. it can be shown that a C^1 boundary (or even less) suffices for the compactness of the imbedding $H^1(\Omega) \hookrightarrow L_2(\Omega)$, Ω bounded).

One of the main points in the following analysis is that we compare an operator with a direct sum of operators. This builds on the elementary

Proposition 5.14. *Let A_1 and A_2 be operators in Hilbert spaces H_1 and H_2, and consider the operator $A = A_1 \oplus A_2$ defined on $H = H_1 \oplus H_2$ (where $\{u_1, u_2\}$ is identified with $u_1 + u_2$, and the norm is $\|u_1 + u_2\|_H = (\|u_1\|^2_{H_1} + \|u_2\|^2_{H_2})^{\frac{1}{2}}$) by:*

$$\begin{aligned} D(A) &= D(A_1) \oplus D(A_2), \\ A(u_1 + u_2) &= A_1 u_1 + A_2 u_2 \quad \text{for } u_1 \in D(A_1), u_2 \in D(A_2). \end{aligned}$$

When the A_j are selfadjoint, so is A. When the A_j are variational, defined from triples (H_j, V_j, s_j), A is variational and is defined from the triple (H, V, s), where $V = V_1 \oplus V_2$, and

$$s(u_1 + u_2, v_1 + v_2) = s_1(u_1, v_1) + s_2(u_2, v_2),$$

denoted $s_1 \oplus s_2$.

When each A_j has the eigenvalues $\lambda_n(A_j)$ and orthogonal eigenvectors $e_n(A_j)$, $n \in M_j$, then A has the eigenvalues and orthogonal eigenvectors

$$\begin{gathered} \{\lambda_n(A_1)\}_{n \in M_1} \cup \{\lambda_m(A_2)\}_{m \in M_2}, \\ \{e_n(A_1) + 0\}_{n \in M_1} \cup \{0 + e_m(A_2)\}_{m \in M_2}; \end{gathered}$$

and vice versa.

In particular, when the A_i are like S in Theorem 5.10, the counting functions satisfy

$$N(t; A_1 \oplus A_2) = N(t; A_1) + N(t; A_2), \text{ for all } t > 0.$$

There are similar statements for the direct sum of a finite number of operators A_j given in Hilbert spaces H_j for $j = 1, \dots, N$; here $A = \bigoplus_{j=1}^N A_j$ on $H = \bigoplus_{j=1}^N H_j$, with sesquilinear form $s = \bigoplus_{j=1}^N s_j$ on $V = \bigoplus_{j=1}^N V_j$ in *the variational case. For operators A_j of the kind treated in Theorem 5.10 one then has the formula:*

$$N(t; \textstyle\bigoplus_{j\le N} A_j) = \sum_{j\le N} N(t; A_j), \text{ for all } t \ge 0. \tag{5.56}$$

Proof. That selfadjointness of A_1 and A_2 implies selfadjointness of A follows from the fact that $A^* = A_1^* \oplus A_2^*$.

For the variational operators, note that coerciveness of the s_j implies coerciveness of s, since (with obvious notation)

$$\begin{aligned} \operatorname{Re} s(u,u) &= \operatorname{Re} s_1(u_1,u_1) + \operatorname{Re} s_2(u_2,u_2) \\ &\ge c_{01}\|u_1\|_{V_1}^2 - k_1\|u_1\|_{H_1}^2 + c_{02}\|u_2\|_{V_2}^2 - k_2\|u_2\|_{H_2}^2 \\ &\qquad \ge \min\{c_{01}, c_{02}\}\|u\|_V^2 - \max\{k_1,k_2\}\|u\|_H^2, \end{aligned}$$

so (H, V, s) defines a variational operator, easily checked to equal $A_1 \oplus A_2$.

The statements on eigenvalues and eigenfunctions are also directly verified. (As usual, eigenvalues are repeated according to multiplicity.)

Sums with more terms are included by induction on the number of terms. $\square$

We shall compare an operator on a general domain Ω with operators on cubes that are contained in Ω, or whose union contains Ω. Then we shall use Theorem 5.11 and Proposition 5.14 in the following two situations:

1°. Let Ω be a bounded open set and let $\{\Omega_j\}_{j\le N}$ be a family of mutually disjoint bounded open sets such that $\bigcup_{j\le N}\Omega_j \subset \Omega$.

Since $H_0^m(\Omega_j)$ is defined as the closure in $H^m(\Omega_j)$ of $C_0^\infty(\Omega_j)$, the extension by 0, followed by restriction to Ω, sends $H_0^m(\Omega_j)$ continuously into $H_0^m(\Omega)$, i.e. we have a continuous injection

$$r_\Omega e_{\Omega_j} \colon H_0^m(\Omega_j) \to H_0^m(\Omega), \text{ for } m \in \mathbb{N}_0.$$

In particular, $r_\Omega e_{\Omega_j}$ injects $L_2(\Omega_j)$ continuously into $L_2(\Omega)$.

Moreover, we can identify:

$$\begin{aligned} \textstyle\bigoplus_{j\le N} L_2(\Omega_j) &\simeq L_2(\textstyle\bigcup_{j\le N}\Omega_j), \text{ closed subspace of } L_2(\Omega) \\ \textstyle\bigoplus_{j\le N} H_0^m(\Omega_j) &\simeq H_0^m(\textstyle\bigcup_{j\le N}\Omega_j), \text{ closed subspace of } H_0^m(\Omega); \end{aligned} \tag{5.57}$$

note here that if Ω_i and Ω_j have a common boundary piece, $H_0^m(\Omega_i \cup \Omega_j)$ is not the same as $H_0^m((\overline{\Omega}_i \cup \overline{\Omega}_j)^\circ)$.

By Theorem 5.13 3°, the injections $H_0^m(\Omega) \hookrightarrow L_2(\Omega)$, $H_0^m(\Omega_j) \hookrightarrow L_2(\Omega_j)$ and $H_0^m(\bigcup\Omega_j) \hookrightarrow L_2(\bigcup\Omega_j)$ are compact when $m > 0$.

Now if $m>0$ and

$$s(u,v)=\sum_{|\alpha|,|\beta|\le m}(a_{\alpha\beta}D^\beta u,D^\alpha v)_{L_2(\Omega)} \tag{5.58}$$

is a symmetric sesquilinear form with smooth bounded coefficients, that is coercive on $H_0^m(\Omega)$ in $L_2(\Omega)$, we can consider the following triples defining variational Dirichlet realizations of $A=\sum_{|\alpha|,|\beta|\le m}D^\alpha a_{\alpha\beta}D^\beta$:

$$\begin{gathered}\big(L_2(\Omega_j),H_0^m(\Omega_j),s\big)\text{ defining }A_{\gamma,\Omega_j}\text{ in }L_2(\Omega_j),\\ \big(L_2(\bigcup\Omega_j),H_0^m(\bigcup\Omega_j),s\big)\text{ defining }A_{\gamma,\bigcup\Omega_j}\text{ in }L_2(\bigcup\Omega_j),\\ \big(L_2(\Omega),H_0^m(\Omega),s\big)\text{ defining }A_{\gamma,\Omega}\text{ in }L_2(\Omega).\end{gathered} \tag{5.59}$$

Using the identifications in (5.57), we have that

$$\textstyle\bigoplus_{j\le N}A_{\gamma,\Omega_j}\text{ identifies with }A_{\gamma,\bigcup_{j\le N}\Omega_j}.$$

Then we can apply Theoren 5.11 with s_1 and s_2 acting like s above and

$$\begin{gathered}H_1=L_2(\bigcup\Omega_j),\quad V_1=H_0^m(\bigcup\Omega_j),\\ H_2=L_2(\Omega),\quad V_2=H_0^m(\Omega).\end{gathered}$$

This gives that

$$\lambda_n(A_{\gamma,\bigcup\Omega_j})=\lambda_n(\textstyle\bigoplus A_{\gamma,\Omega_j})\ge\lambda_n(A_{\gamma,\Omega}),\text{ for all }n\in\mathbb{N}; \tag{5.60}$$

and it implies for the counting functions, in view of (5.56):

$$N(t;A_{\gamma,\bigcup\Omega_j})=\sum_{j\le N}N(t;A_{\gamma,\Omega_j})\le N(t;A_{\gamma,\Omega}),\text{ for all }t\ge 0. \tag{5.61}$$

A geometrical point of view is to regard $A_{\gamma,\bigcup\Omega_j}$ as representing A with *more restrictive* Dirichlet conditions than $A_{\gamma,\Omega}$, since the functions in $D(A_{\gamma,\bigcup\Omega_j})$ must vanish on $\overline{\Omega}\setminus(\bigcup\Omega_j)$ besides on $\partial\Omega$. So (5.61) expresses the following fact: *More restrictive Dirichlet conditions lowers the counting function.* For example, if Ω is divided by a hypersurface Γ' into a disjoint union $\Omega=\Omega_1\cup\Omega_2\cup\Gamma'$, then the functions in $D(A_{\gamma,\Omega_1\cup\Omega_2})$ satisfy an extra Dirichlet condition along Γ', and (5.61) shows the principle:

(I) *The sum of the counting functions for the Dirichlet problems on Ω_1 and Ω_2 is lower than the counting function for the Dirichlet problem on Ω.*

Another special case is when $\Omega \subset \Omega'$ for some other bounded open set Ω'; then

$$\begin{aligned} \lambda_n(A_{\gamma,\Omega}) &\geq \lambda_n(A_{\gamma,\Omega'}), \text{ for all } n \in \mathbb{N}; \\ N(t; A_{\gamma,\Omega}) &\leq N(t; A_{\gamma,\Omega'}), \text{ for all } t \geq 0. \end{aligned} \tag{5.62}$$

Geometrically speaking, we have the principle:

(II) *For the Dirichlet problem, an enlargement of the domain increases the counting function.*

2°. Let $\Omega' = \big(\bigcup_{j\leq N} \overline{\Omega'_j}\big)^\circ$, where the Ω'_j are open cubes with side length 2^{-l} and corners at grid points $2^{-l}(n_1, \dots, n_k) \in 2^{-l}\mathbb{Z}^k$. As shown in Theorem 5.13 4°, the injection $H^m(\Omega') \hookrightarrow L_2(\Omega')$ is compact when $m > 0$.

Here we want to relate $H^m(\Omega')$ to $\bigoplus_{j\in N} H^m(\Omega'_j)$. Clearly, each element $f \in H^m(\Omega')$ defines an element of $\bigoplus H^m(\Omega'_j)$ by the mapping

$$\Phi \colon f \mapsto \{r_{\Omega'_1} f, r_{\Omega'_2} f, \dots, r_{\Omega'_N} f\};$$

this mapping is continuous. Also, if $\Phi(f) = 0$, then $f = 0$ as an element of $L_2(\Omega')$, hence is 0; so Φ is injective. The range space for Φ is simply the set of N-tuples $\{g_1, \dots, g_N\}$ subject to the condition that the traces (up to order $m-1$) of g_i and g_j match whenever Ω'_i and Ω'_j have a face in common, for $i, j = 1, \dots, N$. This condition defines a closed subspace of $\bigoplus H^m(\Omega'_j)$ (since the trace mappings are continuous), so Φ defines an identification of $H^m(\Omega')$ with a closed subspace of $\bigoplus_{j\leq N} H^m(\Omega'_j)$. Moreover, Φ maps $L_2(\Omega')$ bijectively onto $\bigoplus_{j\leq N} L_2(\Omega'_j)$.

Now assume that $m > 0$ and $s(u,v) = \sum_{|\alpha|,|\beta|\leq m} (a_{\alpha\beta} D^\beta u, D^\alpha v)_{L_2(\Omega')}$ (with smooth bounded coefficients) is symmetric and coercive on $H^m(\Omega')$ in $L_2(\Omega')$, and is such that its restrictions to subsets $\Omega'' \subset \Omega'$,

$$s_{\Omega''}(u,v) = \sum_{|\alpha|,|\beta|\leq m} (a_{\alpha\beta} D^\beta u, D^\alpha v)_{L_2(\Omega'')} \tag{5.63}$$

are likewise H^m-coercive there. Let $V_2 = \bigoplus_{j\leq N} H^m(\Omega'_j)$ and consider the form $s_2(u,v) = \bigoplus_{j\leq N} s_{\Omega'_j}(u_j, v_j)$; note that this form on vectors $\Phi u, \Phi v$ gives $s(u,v)$.

Then Theorem 5.11 can be applied with

$$\begin{aligned} H_1 &= L_2(\Omega') = H_2 \, (= \textstyle\bigoplus_{j\leq N} L_2(\Omega'_j)), \\ V_1 &= \Phi H^m(\Omega'), \quad s_1(\Phi u, \Phi v) = s(u,v), \end{aligned}$$

and V_2 and s_2 as already explained. The triple (H_1, V_1, s_1) defines the Neumann realization $A_{\nu,\Omega'}$ of A on Ω'. The triple (H_2, V_2, s_2) defines the direct sum of Neumann realizations $\bigoplus_{j\in N} A_{\nu,\Omega'_j}$. Thus we obtain:

$$\lambda_n(A_{\nu,\Omega'}) \geq \lambda_n(\textstyle\bigoplus_{j\leq N} A_{\nu,\Omega'_j}), \text{ for all } n \in \mathbb{N}. \tag{5.64}$$

For the counting functions, this implies in view of (5.56):

$$N(t;A_{\nu,\Omega'}) \le \sum_{j\le N} N(t;A_{\nu,\Omega'_j}), \text{ for all } t\ge 0. \tag{5.65}$$

In this construction, one could also take more general shapes of the Ω'_j, e.g. finite sums of smooth domains and boxes as in Theorem 5.13 4°, as long as Φ defines a homeomorphism onto a closed subspace of $H^m(\Omega')$. An interesting example is where Γ' is a smooth hypersurface in Ω', dividing it into a disjoint union $\Omega' = \Omega'_1 \cup \Omega'_2 \cup \Gamma'$. Here (5.65) expresses the principle:

(III) *The sum of the counting functions for the Neumann problems on* Ω'_1 *and* Ω'_2 *is higher than the counting function for the Neumann problem on* Ω'.

Note that this goes in the opposite direction than the principle (I) for the Dirichlet problem formulated above!

On could also consider overlapping domains Ω'_j, as long as the intersections are geometrically simple; then the condition for a vector $\{g_1,\dots,g_N\}$ to be a set of restrictions of a function in $H^m(\Omega')$ involves the matching on the overlaps. However, the union of the Ω'_j must *equal* Ω', up to a null-set. So only special geometric shapes of Ω' are allowed. We do *not* for the Neumann problem have a simple principle like (II) above.

Let us finally recall the principle (5.51) that links the Dirichlet and Neumann conditions:

(IV) *Replacing a Dirichlet condition by a Neumann condition increases the counting function.*

To play on these various principles, introducing sub-divisions, is in the more recent literature often called "Dirichlet-Neumann bracketing," cf. e.g. Reed and Simon [R-S 1978, Chapter XIII].

1° and 2° suffice to treat Dirichlet problems on quite general domains, because of the convenient injection property of H_0^m spaces. For Neumann problems, our analysis only gives results for domains Ω' that are unions of cubes. When Ω is a general domain, its intersection with a net of cubes gives some patches near the boundary that are not cubes. The classical method to handle these patches is to make (nonlinear) coordinate transformations that carry them into polygonal shapes (under further hypotheses on the regularity of Ω); this is explained in detail for the case $n=2$ in Volume 1 of Courant and Hilbert's fundamental book on partial differential equations [C-H 1953, pp. 438 ff.], and a method that works for general n is explained in the thesis of L. Sandgren [Sa 1955]. These methods are interesting because they require rather little smoothness of the boundary (but more than the Dirichlet case requires).

When full smoothness is available, we can use Fourier analysis at the boundary, applying other natural points of view. One is to view the Neumann solution operator as a perturbation of the Dirichlet solution operator by an operator that essentially "lives on the boundary" which is of dimension $k-1$, whereby it only interferes with the *remainder term* (the term besides const.$t^{k/2}$) in the asymptotic estimate of the function $N(t)$ (in the way explained e.g. in Grubb [Gr 1984]). We hope to return to this point in connection with the systematic use of Fourier integral methods to construct solution operators later on, and leave out the analysis of Neumann problems for general domains for the time being.

Now let us show how 1° and 2° are used in the case $A = -\Delta$.

Let Ω be a bounded open set. For each integer $l \in \mathbb{N}$, let $\{Q_{j,l}\}_{j=1}^{N_l}$ be an enumeration of the N_l disjoint open cubes with sidelength 2^{-l} and corners at grid points $2^{-l}(n_1, \dots, n_k)$, that are contained wholly in Ω (the inner cubes). Enumerate as $\{Q_{j,l}\}_{j=N_l+1}^{N_l'}$ the open cubes (still with sidelength 2^{-l} and corners at grid points $2^{-l}(n_1, \dots, n_k)$) that have to be added to $\{Q_{j,l}\}_{j=1}^{N_l}$ in order that $\Omega \subset \bigcup_{j=1}^{N_l'} \overline{Q}_{j,l}$ (the whole collection $\{Q_{j,l}\}_{j=1}^{N_l'}$ may be called the outer cubes). Let us denote

$$W_{\mathrm{in},l} = \textstyle\bigcup_{j=1}^{N_l} Q_{j,l}, \quad W_{\mathrm{out},l} = \big(\textstyle\bigcup_{j=1}^{N_l'} \overline{Q}_{j,l}\big)^{\circ}. \tag{5.66}$$

We say (as in [R-S 1978, p. 271]) that Ω is *contented*, when

$$\limsup_{l\to\infty} \operatorname{vol}(W_{\mathrm{in},l}) = \liminf_{l\to\infty} \operatorname{vol}(W_{\mathrm{out},l}) = \operatorname{vol}(\Omega). \tag{5.67}$$

(Here $\operatorname{vol}(M)$ stands for the volume of the set M.)

Theorem 5.15. *For any bounded open subset Ω of $\mathbb{R}^k$, the Dirichlet realization A_γ of $-\Delta$ on Ω has a system of positive eigenvalues going to ∞:*

$$\lambda_1(A_\gamma) \le \lambda_2(A_\gamma) \le \cdots \le \lambda_n(A_\gamma) \le \cdots \to \infty, \tag{5.69}$$

(repeated according to multiplicities), with associated eigenvectors $\{e_n(A_\gamma)\}_{n\in\mathbb{N}}$ forming an orthonormal basis of $L_2(\Omega)$.

If Ω is contented, the counting function $N(t; A_\gamma)$ satisfies

$$N(t, A_\gamma) = (2\pi)^{-k}\omega_k \operatorname{vol}(\Omega) t^{k/2} + o(t^{k/2}),\ \textit{for } t \to \infty. \tag{5.70}$$

If Ω is contained in the closure of the union of inner cubes $Q_{j,l}$ for some l, then

$$N(t, A_\gamma) = (2\pi)^{-k}\omega_k \operatorname{vol}(\Omega) t^{k/2} + O(t^{(k-1)/2}),\ \textit{for } t \to \infty. \tag{5.71}$$

Proof. Since the injection of $H_0^1(\Omega)$ into $L_2(\Omega)$ is compact for any bounded Ω (Theorem 5.13 3°), the first statement holds by Theorem 5.10.

Now let Ω be contented. By (5.14),

$$|N(t;A_{\gamma,Q_0})-2^{-k}\omega_k t^{k/2}|\le Ct^{(k-1)/2},\ \text{for } t\ge 1, \tag{5.72}$$

where C is a constant. A similar estimate holds for $N(t;A_{\nu,Q_0})$. If we replace Q_0 by $\frac{1}{2^l\pi}Q_0=]0,2^{-l}[^k$, the complete system of eigenfunctions for Q_0

$$\sin n_1x_1\cdots\sin n_kx_k,\ \text{with eigenvalues } n_1^2+\cdots+n_k^2,\quad n\in\mathbb{N}^k,$$

is replaced by the complete system of eigenfunctions for $\frac{1}{2^l\pi}Q_0$,

$$\sin 2^l\pi n_1x_1\cdots\sin 2^l\pi n_kx_k,\ \text{with eigenvalues } 2^{2l}\pi^2(n_1^2+\cdots+n_k^2),\ n\in\mathbb{N}^k;$$

so we see that there is the simple relation between the m'th eigenvalues

$$\lambda_m(A_{\gamma,\frac{1}{2^l\pi}Q_0})=2^{2l}\pi^2\lambda_m(A_{\gamma,Q_0}),\ \text{all } m\in\mathbb{N},$$

and hence

$$N(t;A_{\gamma,\frac{1}{2^l\pi}Q_0})=N(\tfrac{t}{2^{2l}\pi^2},A_{\gamma,Q_0}). \tag{5.73}$$

Similarly, we find by consideration of the cosine system (5.15):

$$N(t;A_{\nu,\frac{1}{2^l\pi}Q_0})=N(\tfrac{t}{2^{2l}\pi^2},A_{\nu,Q_0}). \tag{5.74}$$

The formula (5.61) applied to the inner cubes with sidelength 2^{-l} gives, in view of (5.72) and (5.73):

$$\begin{aligned}N(t;A_{\gamma,\Omega})&\ge\sum_{j\le N_l}N(t;A_{\gamma,Q_{j,l}})\ge N_l[2^{-k}\omega_k(\tfrac{t}{2^{2l}\pi^2})^{k/2}-C(\tfrac{t}{2^{2l}\pi^2})^{(k-1)/2}]\\&=\operatorname{vol}(W_{\text{in},l})[(2\pi)^{-k}\omega_kt^{k/2}-C2^l\pi^{1-k}t^{(k-1)/2}],\end{aligned} \tag{5.75}$$

for $t\ge 1$. It follows that

$$\liminf_{t\to\infty}t^{-k/2}N(t;A_{\gamma,\Omega})\ge(2\pi)^{-k}\omega_k\operatorname{vol}(W_{\text{in},l}). \tag{5.76}$$

On the other hand, we have in view of (5.51), (5.56), (5.62) and (5.65), taking $\Omega'=W_{\text{out},l}$,

$$N(t;A_{\gamma,\Omega})\le N(t;A_{\gamma,W_{\text{out},l}})\le N(t;A_{\nu,W_{\text{out},l}})\le\sum_{j\le N_l'}N(t;A_{\nu,Q_{j,l}}).$$

This gives, by (5.72) ff. and (5.74):

$$N(t, A_{\gamma,\Omega}) \leq \sum_{j\leq N_l'} N(t; A_{\gamma,Q_{j,l}}) \leq N_l'[2^{-k}\omega_k(\tfrac{t}{2^{2l}\pi^2})^{k/2} + C(\tfrac{t}{2^{2l}\pi^2})^{(k-1)/2}]$$
$$= \operatorname{vol}(W_{\mathrm{out},l})[(2\pi)^{-k}\omega_k t^{k/2} + C2^l\pi^{1-k}t^{(k-1)/2}], \quad (5.77)$$

for $t \geq 1$, and hence

$$\limsup_{t\to\infty} t^{-k/2}N(t; A_{\gamma,\Omega}) \leq (2\pi)^{-k}\omega_k \operatorname{vol}(W_{\mathrm{out},l}). \quad (5.78)$$

Since (5.76) and (5.78) hold for all l, we can conclude in view of (5.67) that

$$\lim_{t\to\infty} t^{-k/2}N(t; A_{\gamma,\Omega}) = (2\pi)^{-k}\omega_k \operatorname{vol}(\Omega), \quad (5.79)$$

which can also be written as (5.70).

The last statement in the theorem is concerned with a case where the set of inner cubes $Q_{j,l}$, $j \leq N_l$, equals the set of outer cubes. In particular,

$$\operatorname{vol}(W_{\mathrm{in},l}) = \operatorname{vol}(W_{\mathrm{out},l}) = \operatorname{vol}(\Omega). \quad (5.80)$$

Then (5.71) follows from directly from (5.75) and (5.77). □

Theorem 5.16. *When $\Omega \subset \mathbb{R}^k$ is such that the injection $H^1(\Omega) \hookrightarrow L_2(\Omega)$ is compact (in particular, when Ω is a finite union of smooth bounded open sets and open cubes), the Neumann realization A_ν of $-\Delta$ on Ω has a system of nonnegative eigenvalues going to ∞:*

$$\lambda_1(A_\nu) \leq \lambda_2(A_\nu) \leq \cdots \leq \lambda_n(A_\nu) \leq \cdots \to \infty, \quad (5.81)$$

(repeated according to multiplicities), with associated eigenvectors $\{e_n(A_\nu)\}_{n\in\mathbb{N}}$ forming an orthonormal basis of $L_2(\Omega)$.

If Ω is contained in the closure of the union of inner cubes $Q_{j,l}$ for some l, then

$$N(t, A_\nu) = (2\pi)^{-k}\omega_k \operatorname{vol}(\Omega)t^{k/2} + O(t^{(k-1)/2}), \text{ for } t \to \infty. \quad (5.82)$$

Proof. The first statement follows from Theorem 5.10 (cf. also Theorem 5.13 4°).

Now consider an Ω satisfying the special hypothesis, so that (5.80) holds. On one hand, we have in view of (5.51), (5.61), (5.72) and (5.73):

$$N(t; A_{\nu,\Omega}) \geq N(t; A_{\gamma,\Omega}) \geq \sum_{j\leq N_l} N(t; A_{\gamma,Q_{j,l}})$$
$$\geq N_l[2^{-k}\omega_k(\tfrac{t}{2^{2l}\pi^2})^{k/2} - C(\tfrac{t}{2^{2l}\pi^2})^{(k-1)/2}]$$
$$= \operatorname{vol}(W_{\mathrm{in},l})[(2\pi)^{-k}\omega_k t^{k/2} - C2^l\pi^{1-k}t^{(k-1)/2}], \quad (5.83)$$

for $t \geq 1$. On the other hand, (5.65) and (5.74) give:

$$N(t, A_{\nu,\Omega}) \leq \sum_{j \leq N_l} N(t; A_{\nu,Q_{j,l}}) \leq N_l[2^{-k}\omega_k(\tfrac{t}{2^{2l}\pi^2})^{k/2} + C(\tfrac{t}{2^{2l}\pi^2})^{(k-1)/2}]$$
$$= \operatorname{vol}(W_{\text{out},l})[(2\pi)^{-k}\omega_k t^{k/2} + C2^l\pi^{1-k}t^{(k-1)/2}], \quad (5.84)$$

for $t \geq 1$. Since (5.80) holds, this proves (5.81). $\square$

The finer estimates in (5.71) and (5.82), that we have here obtained for domains that are unions of congruent cubes, can be shown for much more general domains by use of deep theories. In fact, the question of their validity was open for general smooth domains up until the seventies, where the fundamental works of L. Hörmander [H 1968], [H 1971], brought new methods into the picture (Fourier integral operators).

The asymptotic estimates, we have shown above, are equivalent with asymptotic estimates of eigenvalues in the following way:

Theorem 5.17. *Let H be a Hilbert space, and let A be a selfadjoint positive operator in H with compact inverse. Let $\{\lambda_n\}_{n\in\mathbb{N}}$ be the eigenvalue sequence (arranged as in* (5.41)*) and $N(t)$ the counting function; $N(t) = \#\{\text{eigenvalues} \leq t\}$. Let $C > 0$, $\alpha > \beta > 0$.*

1° *The counting function satisfies:*

$$N(t) = Ct^\alpha + o(t^\alpha) \quad \text{for } t \to \infty, \tag{5.85}$$

if and only if the eigenvalue sequence satisfies:

$$\lambda_n = C^{-1/\alpha}n^{1/\alpha} + o(n^{1/\alpha}) \quad \text{for } n \to \infty. \tag{5.86}$$

2° *The counting function satisfies:*

$$N(t) = Ct^\alpha + O(t^\beta) \quad \text{for } t \to \infty, \tag{5.87}$$

if and only if the eigenvalue sequence satisfies, with $\gamma = (1 - \alpha + \beta)/\alpha$:

$$\lambda_n = C^{-1/\alpha}n^{1/\alpha} + O(n^\gamma) \quad \text{for } n \to \infty. \tag{5.88}$$

Proof. Note that the functions $t \mapsto N(t)$ and $n \mapsto \lambda_n$ are in some sense inverses of one another. More precisely, let $\{n_j\}_{j\in\mathbb{N}}$ be the subsequence of n where the inequality $\lambda_n \leq \lambda_{n+1}$ is strict, i.e.

$$\lambda_{n_j} < \lambda_{n_j+1}, \tag{5.89}$$

and set $n_0 = 0 = \lambda_{n_0}$; then $N(t)$ takes the value n_j on the interval $[\lambda_{n_j}, \lambda_{n_j+1}[= [\lambda_{n_j}, \lambda_{n_{j+1}}[$. Here $N(t)$ jumps from n_{j-1} to n_j at $\lambda_{n_j} > 0$, and is continuous from the right. In particular,

$$N\colon \{\lambda_{n_0}, \lambda_{n_1}, \lambda_{n_2} \dots\} \to \{n_0, n_1, n_2, \dots\} \text{ bijectively,}$$
$$\text{with } N(\lambda_{n_j}) = n_j. \quad (5.90)$$

We can consider, along with $N(t)$, the auxiliary counting function

$$N_<(t) = \#\{\text{eigenvalues} < t\}; \tag{5.91}$$

for t not equal to an eigenvalue it coincides with $N(t)$, and at λ_{n_j} it jumps from n_{j-1} to n_j, being continuous from the left. Then for any λ_n,

$$N_<(\lambda_n) \le n \le N(\lambda_n). \tag{5.92}$$

For, any n belongs to an interval $[n_j + 1, n_{j+1}]$, where $\lambda_{n_j+1} = \lambda_n = \lambda_{n_{j+1}}$, so $N(\lambda_n) = N(\lambda_{n_{j+1}}) = n_{j+1}$, and $N_<(\lambda_n) = N_<(\lambda_{n_j+1}) = n_j$.

Note that $N_<(t)$ differs very little from $N(t)$; in fact

$$N_<(t) = \lim_{s \nearrow t} N(s), \quad N(t) = \lim_{s \searrow t} N_<(s). \tag{5.93}$$

Thus (5.85) resp. (5.87) implies the same property for $N_<(t)$.

1°. Note that (5.85) means that

$$t^{-\alpha} N(t) \to C, \text{ for } t \to \infty; \tag{5.94}$$

and as just observed, then also $t^{-\alpha} N_<(t) \to C$. For $t = \lambda_n$, we have in view of (5.92),

$$\lambda_n^{-\alpha} N_<(\lambda_n) \le \lambda_n^{-\alpha} n \le \lambda_n^{-\alpha} N(\lambda_n),$$

so it follows that $\lambda_n^{-\alpha} n \to C$ for $n \to \infty$, and this shows (5.86).

For the converse, we note that any t satisfies

$$\lambda_{n_j} \le t < \lambda_{n_j+1} \tag{5.95}$$

for a uniquely determined j. Here $N(t) = n_j$, so it follows that

$$\lambda_{n_j+1}^{-\alpha} n_j < t^{-\alpha} N(t) \le \lambda_{n_j}^{-\alpha} n_j. \tag{5.96}$$

Now $t \to \infty \implies j \to \infty$, and when (5.86) holds, the first and the last expressions in (5.96) converge to C for $j \to \infty$, so it follows that $t^{-\alpha} N(t) \to C$ for $t \to \infty$, showing (5.85). This ends the proof of 1°.

2°. Assume (5.87); as noted above, it also holds for $N_<(t)$, so, for some constant C_1,

$$-C_1t^\beta \le N_<(t) - Ct^\alpha \le N(t) - Ct^\alpha \le C_1t^\beta \text{ for all } t \ge 0.$$

Taking $t = \lambda_n$, we then have in view of (5.92):

$$-C_1\lambda_n^\beta \le n - C\lambda_n^\alpha \le C_1\lambda_n^\beta \text{ for all } n \ge 1. \tag{5.97}$$

As already shown under 1°, $n\lambda_n^{-\alpha} \to C$ for $n \to \infty$, so there is a rough estimate

$$C_2n^{1/\alpha} \le \lambda_n \le C_3n^{1/\alpha} \text{ for } n \ge C_4,$$

with positive constants C_j.

We now find from the right inequality in (5.97), for large n:

$$\begin{aligned}\lambda_n &\ge (C^{-1}n - C^{-1}C_1\lambda_n^\beta)^{1/\alpha}\\ &\ge C^{-1/\alpha}n^{1/\alpha}(1 - C_1n^{-1}(C_3n^{1/\alpha})^\beta)^{1/\alpha}\\ &\ge C^{-1/\alpha}n^{1/\alpha}(1 - C_5n^{-1+\beta/\alpha}) = C^{-1/\alpha}n^{1/\alpha} - C_6n^\gamma;\end{aligned} \tag{5.98}$$

it is used here that $(1-\varepsilon)^{1/\alpha} \ge 1 - C_7\varepsilon$ for $\varepsilon \in]0,1]$. Similarly, the left inequality gives, for large n:

$$\begin{aligned}\lambda_n &\le (C^{-1}n + C^{-1}C_1\lambda_n^\beta)^{1/\alpha}\\ &\le C^{-1/\alpha}n^{1/\alpha}(1 + C_1n^{-1}(C_3n^{1/\alpha})^\beta)^{1/\alpha}\\ &\le C^{-1/\alpha}n^{1/\alpha}(1 + C_8n^{-1+\beta/\alpha}) = C^{-1/\alpha}n^{1/\alpha} + C_9n^\gamma;\end{aligned} \tag{5.99}$$

using that $(1+\varepsilon)^{1/\alpha} \le 1 + C_{10}\varepsilon$ for $\varepsilon \in]0,1]$. (5.98) and (5.99) together show (5.88).

Finally, let (5.88) hold; i.e.,

$$-C_1'n^\gamma \le \lambda_n - C^{-1/\alpha}n^{1/\alpha} \le C_1'n^\gamma, \tag{5.100}$$

for some $C_1' \ge 0$. Note that $\gamma < 1/\alpha$. As shown under 1°, $t^{-\alpha}N(t) \to C$ then, so one has a rough estimate:

$$C_2't^\alpha \le N(t) \le C_3't^\alpha \text{ for } t \ge C_4', \tag{5.101}$$

with positive constants C_j'. Let $t = \lambda_{n_j}$ so that $N(t) = n_j$; cf. (5.90). The right inequality in (5.100) gives

$$\begin{aligned}N(t) = n_j &\ge (C^{1/\alpha}\lambda_{n_j} - C^{1/\alpha}C_1'n_j^\gamma)^\alpha\\ &= Ct^\alpha(1 - t^{-1}C_1'N(t)^{(1-\alpha+\beta)/\alpha})^\alpha\\ &\ge Ct^\alpha(1 - t^{-1}C_1'{C_3'}^\gamma t^{1-\alpha+\beta})^\alpha\\ &\ge Ct^\alpha - C_5't^\beta\end{aligned}$$

(as above), and similarly the left inequality implies $N(t) \le Ct^\alpha + C_6' t^\beta$. This shows the desired estimates for $t = \lambda_{n_j}$;

$$Ct^\alpha - C_5' t^\beta \le N(t) \le Ct^\alpha + C_6' t^\beta, \text{ for } t = \lambda_{n_j}. \tag{5.102}$$

For general t we use (5.95) ff., that leads to

$$\lambda_{n_j+1}^{-\alpha} N(\lambda_{n_j}) - C \le t^{-\alpha} N(t) - C \le \lambda_{n_j}^{-\alpha} N(\lambda_{n_j}) - C. \tag{5.103}$$

Here we need to estimate $\lambda_{n_j+1} - \lambda_{n_j}$. In view of (5.100) and (5.101), and the fact that (by the mean value theorem, with $\theta(x) \in]0,1[$)

$$\frac{(x+1)^{1/\alpha} - x^{1/\alpha}}{1} = \tfrac{1}{\alpha}(x+\theta(x))^{1/\alpha - 1} \le C_0'' x^{1/\alpha - 1}$$

for large x, we have that

$$\begin{aligned} 0 \le \lambda_{n_j+1} - \lambda_{n_j} &\le C^{-1/\alpha}((n_j+1)^{1/\alpha} - n_j^{1/\alpha}) + C_1'((n_j+1)^\gamma + n_j^\gamma) \\ &\le C^{-1/\alpha} C_0'' N(t)^{1/\alpha - 1} + C_1'((N(t)+1)^\gamma + N(t)^\gamma) \le C_1'' t^{1-\alpha+\beta} + C_2'', \end{aligned}$$

for large t. Then

$$\begin{aligned} \lambda_{n_j+1}^{-\alpha} &= \lambda_{n_j}^{-\alpha}(1 + \lambda_{n_j}^{-1}(\lambda_{n_j+1} - \lambda_{n_j}))^{-\alpha} \\ &\ge \lambda_{n_j}^{-\alpha}(1 - C_3'' \lambda_{n_j}^{-1}(C_1'' t^{1-\alpha+\beta} + C_2'')). \end{aligned} \tag{5.104}$$

The right inequalities in (5.103) and (5.102) give, by use of the rough estimates,

$$\begin{aligned} t^{-\alpha} N(t) - C \le \lambda_{n_j}^{-\alpha} N(\lambda_{n_j}) - C &\le C_6' \lambda_{n_j}^{\beta-\alpha} \le C_4'' n_j^{(\beta-\alpha)/\alpha} \\ &= C_4'' N(t)^{(\beta-\alpha)/\alpha} \le C_5'' t^{\beta-\alpha}. \end{aligned}$$

The left inequalities give, by use of (5.104) and the rough estimates,

$$\begin{aligned} t^{-\alpha} N(t) - C &\ge \lambda_{n_j+1}^{-\alpha} N(\lambda_{n_j}) - C \\ &\ge \lambda_{n_j}^{-\alpha}(1 - C_3'' \lambda_{n_j}^{-1}(C_1'' t^{1-\alpha+\beta} + C_2''))N(\lambda_{n_j}) - C \\ &\ge -C_5' \lambda_{n_j}^{\beta-\alpha} - C_3'' \lambda_{n_j}^{-\alpha-1}(C_1'' t^{1-\alpha+\beta} + C_2'') N(\lambda_{n_j}) \\ &\ge -C_6'' t^{\beta-\alpha} - C_7'' t^{-\alpha-1}(C_1'' t^{1-\alpha+\beta} + C_2'') t^\alpha \\ &\ge -C_8'' t^{\beta-\alpha}, \end{aligned}$$

for large t. This shows (5.87) for general t, and ends the proof. $\square$

For the Dirichlet and Neumann problems of $-\Delta$ considered in Theorems 5.15 and 5.16, $\alpha = k/2$, $\beta = (k-1)/2$ and $\gamma = 1/k$, so here

$$\begin{aligned} N(t) &= Ct^{k/2} + O(t^{(k-1)/2}) \text{ for } t \to \infty \iff \\ \lambda_n &= C^{-2/k} n^{2/k} + O(n^{1/k}) \text{ for } n \to \infty. \end{aligned} \tag{5.105}$$

The other counting function $N_<(t)$, cf. (5.91), is used instead of $N(t)$ in many texts. (5.93) shows that they differ very little and, as we saw in the proof, typical estimates for $N(t)$ hold for $N_<(t)$ as well. Also an interesting precise estimate such as (for $-\Delta$)

$$N(t) = Ct^{k/2} + C' t^{(k-1)/2} + o(t^{(k-1)/2}), \text{ for } t \to \infty, \tag{5.106}$$

that has been shown in recent years, carries over to $N_<(t)$ (and vice versa).

Note that where the two counting functions differ, the difference equals the multiplicity of the eigenvalue in question:

$$N(\lambda_{n_j}) - N_<(\lambda_{n_j}) = n_j - n_{j-1} = \nu(\lambda_{n_j}),$$

the multiplicity of λ_{n_j}. It satisfies the same estimate as the remainder term for $N(t)$:

$$\begin{aligned} (5.85) &\implies \nu(\lambda_{n_j}) = o(\lambda_{n_j}^{\alpha}) = o(n_j), \\ (5.87) &\implies \nu(\lambda_{n_j}) = O(\lambda_{n_j}^{\beta}) = O(n_j^{\beta/\alpha}). \end{aligned} \tag{5.107}$$