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Weakly parametric pseudodifferential operators and Atiyah-Patodi-Singer boundary problems

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This paper introduces a class of pseudodifferential operators depending on a parameter in a particular way. The main application is a complete expansion of the trace of the resolvent of a Dirac-type operator with nonlocal boundary conditions of the kind introduced by Atiyah, Patodi, and Singer [APS]. This extends the partial expansion in [G2] to a complete one, and extends the complete expansion in [GS1] to the case where the Dirac operator does not have a product structure near the boundary. A secondary application is to obtain a complete expansion of the resolvent of a ψdo on a compact manifold, essentially reproving a result of Agranovich [Agr]. The resolvent expansion yields immediately an expansion of the trace of the heat kernel, and determines the singularities of the zeta function; moreover, a pseudodifferential factor can be allowed.

A major motive for these expansions is to obtain index formulas for elliptic operators; there are many such applications in the physics and geometry literature. The index formula comes from one particular term in the expansion, but each term is a spectral invariant, and they have been used for other purposes as well as for the index. In particular, Branson and Gilkey have a number of papers (e.g. [BG] and [Gi]) analyzing these invariants, and drawing geometric consequences.

Interest in the asymptotic behavior of the resolvent goes back to Carleman [C]. More recently, Agmon [Agm] developed it extensively for analytic applications; he introduced the fundamental idea of treating the resolvent parameter essentially as another cotangent variable. This idea was developed in [S1] to analyze the singularities of the zeta function of an elliptic ψdo on a compact manifold, and in [S3] to analyze the resolvent of a differential operator with differential boundary conditions. The technique works smoothly for differential operators, producing so-called local invariants, integrals over the underlying

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manifold of densities defined locally by the symbol of the operator involved. But in the pseudodifferential case it only goes so far. In particular, [APS] identifies an interesting nonlocal term in the expansion, contributing to the index formula. Grubb [G2] has studied carefully how far the Agmon approach does go, in a framework of pseudodifferential boundary problems, obtaining an expansion up to and including the first nonlocal term. The present paper modifies the technique, thus yielding the complete expansion, with a full sequence of nonlocal terms, and logarithmic terms as noticed by [DG] in the case of a ψdo on a compact manifold.

A simple example illustrates the modification. Suppose that $a(x,\xi)$ is the leading symbol of a first-order elliptic system A. Then the first term in the ψdo expansion of the resolvent $(A-\lambda)^{-1}$ has symbol $(a-\lambda)^{-1}$. If a is a polynomial in ξ, then each ξ-derivative of $(a-\lambda)^{-1}$ increases the rate of decay as $\lambda\to\infty$; this is an important property of the class of ψdo's with parameter as studied e.g. in Shubin [Sh]. But if a is a general ψdo symbol, homogeneous only for $|\xi|\geqq 1$ (when taken to be C^∞ for all ξ), then the ξ-derivatives of $(a-\lambda)^{-1}$ do not produce decay greater than λ^{-2}. Still, $(a-\lambda)^{-1}$ does have an expansion in decreasing powers of λ, with coefficients which have *increasing* growth as $|\xi|\to\infty$. It is this sort of expansion which we require of our symbols, and we call them *weakly parametric* ψdo's, in contrast to those in [Sh].

It is appropriate here to point out some deficiencies in the earlier paper [S1] on this topic. A technical problem was corrected in [S3]. Then [DG] noted that certain singularities occurring in the case of ψdo's had been overlooked. In [S1], p. 290, it was stated that the residues of the zeta function $\mathrm{Tr}(A^s)$ vanish for $s=1,2,\ldots$. This is true only for differential operators.

The paper is divided into 3 sections and an appendix. In the first section we define and study our class of weakly parametric ψdo's, for a compact manifold without boundary. Most proofs here are straightforward, and may be skipped on a first reading. Section 2 includes a general expansion theorem for this type of operator (Theorem 2.1). The proof shows how logarithmic terms and nonlocal terms arise in the expansion. The theorem is applied to the case of the resolvent of an elliptic ψdo on a compact manifold. Section 3 treats the case of an APS operator P_B, a Dirac-type differential operator P with ψdo boundary conditions (including those of [APS]). We construct the resolvent of P_B. Since the operator in the interior is differential, the associated operators connecting the interior with the boundary are relatively simple (so-called strongly polyhomogeneous). Moreover, the trace calculation can be reduced to the boundary, where the theory from Sections 1 and 2 applies. One of the main results (Theorem 3.13) is a complete description of the zeta function pole structure, and a full heat trace expansion:

$$\Gamma(s)\mathrm{Tr}(\varphi(P_B{}^*P_B)^{-s})\sim\frac{-c'}{s}+\frac{\tilde a_0}{s-\frac{n}{2}}+\sum_{j=1}^{\infty}\frac{\tilde a_j+\tilde b_j}{s-\frac{n-j}{2}}+\sum_{j=0}^{\infty}\Big(\frac{\tilde c_j}{(s+\frac{j}{2})^2}+\frac{\tilde c_j'}{s+\frac{j}{2}}\Big);$$

$$\mathrm{Tr}(\varphi e^{-tP_B{}^*P_B})\sim\tilde a_0t^{-\frac{n}{2}}+\sum_{j=1}^{\infty}(\tilde a_j+\tilde b_j)t^{\frac{j-n}{2}}+\sum_{j=0}^{\infty}\big(-\tilde c_jt^{\frac{j}{2}}\log t+\tilde c_j't^{\frac{j}{2}}\big).$$

The primed coefficients are in general globally defined, while the others are local. The appendix gives a selfcontained presentation of the necessary results from the boundary operator calculus. The main results were announced in [GS2].

One can moreover set up a general (more complicated) calculus of weakly polyhomogeneous boundary operators, to which our operators belong; but since it is not needed for the present trace formulas, it is not included in this paper.

Contents

1. Weakly parametric ψdo's

1.1. Parameter-dependent symbols

By $S^m(\mathbb{R}^\nu, \mathbb{R}^n)$ we denote the standard ψdo symbol space consisting of the functions $p(x,\xi) \in C^\infty(\mathbb{R}^\nu \times \mathbb{R}^n)$ such that $\partial_x^\beta \partial_\xi^\alpha p$ is $O(\langle\xi\rangle^{m-|\alpha|})$ for all $\alpha \in \mathbb{N}^n$, $\beta \in \mathbb{N}^\nu$; here $\langle\xi\rangle = (1+|\xi|^2)^{\frac{1}{2}}$ and $\mathbb{N} = \{\text{integers} \geqq 0\}$. The rules of calculus for this space are well-known, see e.g. [S4], [H2,3], [T], [Sh]. A symbol $p \in S^m(\mathbb{R}^\nu, \mathbb{R}^n)$ is called "classical polyhomogeneous" (of degree m) if it has an expansion $p \sim \sum_{j\in\mathbb{N}} p_j$, where the p_j are homogeneous in ξ of degree $m-j$ for $|\xi| \geqq 1$, and $p - \sum_{j<J} p_j \in S^{m-J}(\mathbb{R}^\nu, \mathbb{R}^n)$. Besides the usual notation $\langle\xi\rangle$ it will be convenient to introduce a function $t \mapsto [t]$ such that

$$(1.1) \quad [t] \in C^\infty(\mathbb{R}),\ [t] = |t| \text{ for } |t| \geqq 1,\ [t] \in \left[\tfrac{1}{2}, 1\right] \text{ for } |t| \leqq 1,$$
$$[t] = \tfrac{1}{2} \text{ for } |t| \leqq \tfrac{1}{4}; \quad \text{then we also denote } [|\xi|] = [\xi], \text{ for } \xi \in \mathbb{R}^n.$$

In the definition of $S^m(\mathbb{R}^\nu, \mathbb{R}^n)$, $[\xi]$ can be used instead of $\langle\xi\rangle$ and has the advantage of being homogeneous for $|\xi| \geqq 1$. Further below, we also include a real or complex parameter μ; then we denote

$$(1.2) \qquad |(\xi,\mu)| = (|\xi|^2 + |\mu|^2)^{\frac{1}{2}} = |\xi,\mu|, \quad [|(\xi,\mu)|] = [\xi,\mu].$$

Since we are interested in questions local in x, p need not always be defined throughout $\mathbb{R}^\nu$, and "big O" is uniform in ξ, but not necessarily in x (only locally so).

We shall now define a class of symbols p depending on a parameter μ varying in a sector $\Gamma \subset \mathbb{C} \setminus \{0\}$. ($\Gamma$ is of the form $\{\lambda = re^{i\theta} \mid r > 0, \theta \in I\}$, I a subset of $[0, 2\pi]$, and Γ is closed in $\mathbb{C} \setminus \{0\}$ when I is closed.) It is the behavior for $|\mu| \to \infty$ that is important here, and we often describe it in terms of the behavior of $p(x,\xi,\frac{1}{z})$ for $z \to 0$, $\frac{1}{z} = \mu \in \Gamma$. For brevity of notation, we write $\partial_z^j p(x,\xi,\frac{1}{z})$ (or just $\partial_z^j p$) for the j'th z-derivative of the composite function $z \mapsto p(x,\xi,\frac{1}{z})$.

Definition 1.1. *Let n and ν be positive integers, and let m and $d \in \mathbb{R}$. Let Γ be a sector in $\mathbb{C}\setminus\{0\}$. The space $S^{m,0}(\mathbb{R}^\nu,\mathbb{R}^n,\Gamma)$ consists of the functions $p(x,\xi,\mu)\in C^\infty(\mathbb{R}^\nu\times\mathbb{R}^n\times\Gamma)$ that are holomorphic in $\mu\in\overset{\circ}{\Gamma}$ for $|\xi,\mu|\geqq\varepsilon$ (some $\varepsilon>0$) and satisfy, for all $j\in\mathbb{N}$,*

$$\partial_z^j p(\cdot,\cdot,\tfrac{1}{z}) \text{ is in } S^{m+j}(\mathbb{R}^\nu,\mathbb{R}^n) \quad \text{for } \tfrac{1}{z}\in\Gamma,$$
$$\text{with uniform estimates for } |z|\leqq 1, \tfrac{1}{z}\in \text{closed subsectors of } \Gamma.$$

Moreover, we set $S^{m,d}=\mu^d S^{m,0}$; that is, $S^{m,d}(\mathbb{R}^\nu,\mathbb{R}^n,\Gamma)$ consists of the functions p (holomorphic in $\mu\in\overset{\circ}{\Gamma}$ for $|\xi,\mu|\geqq\varepsilon$) such that for all $j\in\mathbb{N}$,

$$\partial_z^j(z^d p(\cdot,\cdot,\tfrac{1}{z})) \text{ is in } S^{m+j}(\mathbb{R}^\nu,\mathbb{R}^n) \quad \text{for } \tfrac{1}{z}\in\Gamma,$$
$$\text{with uniform estimates for } |z|\leqq 1, \tfrac{1}{z}\in \text{closed subsectors of } \Gamma.$$

We call these symbols weakly parametric. Polyhomogeneous subclasses are considered below in Definition 1.10 ff. The symbol spaces are Fréchet spaces with systems of seminorms defined in the usual way:

$$p\mapsto \sup\{\langle\xi\rangle^{-m-j+|\alpha|}|\partial_x^\beta\partial_\xi^\alpha\partial_z^j(z^d p(x,\xi,\tfrac{1}{z}))| \;|\; x\in K, \xi\in\mathbb{R}^n, \tfrac{1}{z}\in\Gamma' \text{ with } |z|\leqq 1\},$$
$$\text{for } \alpha\in\mathbb{N}^n,\ \beta\in\mathbb{N}^\nu,\ j\in\mathbb{N},\ K \text{ compact}\subset\mathbb{R}^\nu,\ \Gamma' \text{closed}\subset\Gamma. \tag{1.3}$$

Example 1.2. For symbols independent of μ, one clearly has that $S^m(\mathbb{R}^\nu,\mathbb{R}^n) \subset S^{m,0}(\mathbb{R}^\nu,\mathbb{R}^n,\mathbb{C})$. On the other hand, if $f(\mu)$ is meromorphic in $\mathbb{C}$ and of order d at ∞, then $f\in S^{0,d}(\mathbb{R}^\nu,\mathbb{R}^n,\Gamma)$ for any Γ in the sector excluding poles of f.

Lemma 1.3. *If $f(x,\xi)\in S^m(\mathbb{R}^\nu,\mathbb{R}^n)$ with $m\leqq 0$, then $p(x,\xi,\mu)=f(x,\xi/\mu)$ is in $S^{0,0}(\mathbb{R}^\nu,\mathbb{R}^n,\mathbb{R}_+)$.*

Proof. In this case, $\overset{\circ}{\Gamma}$ is empty, so p need not be holomorphic. We can ignore the variable x. Note first that when $f\in S^m$, then a fortiori $f\in S^0$, so we can assume that $m=0$. Write $\frac{1}{\mu}=z\in\mathbb{R}_+$. We must show that $\partial_z^j f(z\xi)\in S^j$, uniformly for $0<z\leqq 1$.

For $j=0$, this follows from the estimates

$$\begin{aligned}|\partial_\xi^\alpha f(z\xi)| &= z^{|\alpha|}|\partial_\eta^\alpha f(\eta)|_{\eta=z\xi}| \leqq C_\alpha z^{|\alpha|}(1+|z\xi|^2)^{-|\alpha|/2}\\ &= C_\alpha(z^{-2}+|\xi|^2)^{-|\alpha|/2}\leqq C_\alpha(1+|\xi|^2)^{-|\alpha|/2},\end{aligned}$$

that hold with C_α independent of z, since $z\leqq 1$. For $j>0$ we observe that

$$\partial_z^j f(z\xi)=\sum_{|\beta|=j} C_\beta\xi^\beta\partial_\eta^\beta f(\eta)|_{\eta=z\xi}.$$

Here $\partial_\eta^\beta f(\eta) \in S^{-|\beta|} \subset S^0$, so the first part of the proof shows that $\partial_\eta^\beta f(\eta)|_{\eta=z\xi}$ is uniformly in S^0 for $z \leqq 1$. Multiplication by ξ^β ($|\beta| = j$) then gives a symbol that is uniformly in S^j for $z \leqq 1$. □

Example 1.4. Suppose that $p(x, \xi, \mu) \in C^\infty(\mathbb{R}^\nu \times \mathbb{R}^n \times \overline{\mathbb{R}}_+)$ and p is homogeneous of degree 0 in (ξ, μ) for $|\xi, \mu| \geqq 1$. Then $p \in S^{0,0}(\mathbb{R}^\nu, \mathbb{R}^n, \mathbb{R}_+)$. To see this, note first that the (ξ, μ)-derivatives of order k are homogeneous of degree $-k$ for $|\xi, \mu| \geqq 1$, and continuous on $|\xi, \mu| \leqq 1$, even at $\mu = 0$; hence they are $O([\xi, \mu]^{-k})$. In particular,

$$\partial_x^\beta \partial_\xi^\alpha p(x, \xi, \mu_0) = O([\xi, \mu_0]^{-|\alpha|}) = O([\xi]^{-|\alpha|}), \text{ for each } \mu_0 \geqq 0,$$

so p is in S^0 for each fixed μ_0. It remains to show the required uniformity for $\mu \geqq 1$ (i.e $z \leqq 1$). Here $p(x, \xi, \mu) = p(x, \xi/\mu, 1)$ by the homogeneity, so Lemma 1.3 applies with $f(x, \xi) = p(x, \xi, 1)$.

Lemma 1.5. *For the spaces introduced in Definition 1.1 the following maps are continuous:*

$$(1.4)\quad \begin{aligned} &\text{(i)} && \partial_\xi^\alpha : S^{m,d} \to S^{m-|\alpha|,d}, \\ &\text{(ii)} && \partial_x^\beta : S^{m,d} \to S^{m,d}, \\ &\text{(iii)} && z^k : S^{m,d} \xrightarrow{\sim} S^{m,d-k}, \\ &\text{(iv)} && \partial_z : S^{m,0} \to S^{m+1,0}, \\ &\text{(v)} && \partial_z : S^{m,d} \to S^{m+1,d} + S^{m,d+1}, \text{ when } d \neq 0. \end{aligned}$$

Proof. The statements (i)–(iv) follow immediately from the definition. For (v), note that applying ∂_z on $S^{m,d}$ is equivalent to applying $\partial_z z^{-d}$ on $S^{m,0}$, and here

$$\partial_z(z^{-d} p) = z^{-d} \partial_z p - d z^{-d-1} p. \quad \square$$

Example 1.2 and (v) show that ∂_z does not simply map $S^{m,d}$ into $S^{m+1,d}$ when $d \neq 0$.

Lemma 1.6. $p \in S^{m,d}(\mathbb{R}^\nu, \mathbb{R}^n, \Gamma)$, $q \in S^{m',d'}(\mathbb{R}^{\nu'}, \mathbb{R}^n, \Gamma)$ *imply* $p(x, \xi, \mu) \cdot q(y, \xi, \mu) \in S^{m+m', d+d'}(\mathbb{R}^{\nu+\nu'}, \mathbb{R}^n, \Gamma)$.

Proof. By the Leibniz formula,

$$\partial_z^j (z^{d+d'} pq) = \sum_{0 \leqq k \leqq j} \binom{j}{k} \partial_z^k (z^d p) \partial_z^{j-k} (z^{d'} q),$$

which belongs to $S^{m+m'+j}$ uniformly in z ($|z| \leqq 1$, $\frac{1}{z} \in \Gamma$), since $\partial_z^k (z^d p) \in S^{m-k}$ and $\partial_z^{j-k}(z^{d'} q) \in S^{m'-j+k}$ uniformly, and $S^{m-k} S^{m'-j+k} \subset S^{m+m'-j}$. □

Lemma 1.7. $S^{m,d} \subset S^{m',d'}$ (*with continuous injection*) *if* $m \leqq m'$ *and* $d' - d \in \mathbb{N}$.

Proof. Clearly, $S^{m,0} \subset S^{m',0}$ for $m \leqq m'$, and $zS^{m,0} \subset S^{m,0}$ since $\partial_z^j(zp) = j\partial_z^{j-1}p + z\partial_z^j p$ is bounded in $S^{m+j-1} + S^{m+j} \subset S^{m+j}$. Thus when $d' - d \in \mathbb{N}$,

$$S^{m,d} = z^{-d}S^{m,0} = z^{-d'}(z^{d'-d}S^{m,0}) \subset z^{-d'}S^{m,0} \subset z^{-d'}S^{m',0} = S^{m',d'}. \qquad \square$$

We denote

$$\bigcup_{m\in\mathbb{R}} S^{m,d} = S^{\infty,d}, \qquad \bigcap_{m\in\mathbb{R}} S^{m,d} = S^{-\infty,d}. \tag{1.5}$$

Definition 1.8. *Let p_j ($j \in \mathbb{N}$) be a sequence of symbols in $S^{m_j,d}$, where $m_j \searrow -\infty$. Then*

$$p \sim \sum_{j\in\mathbb{N}} p_j \text{ in } S^{\infty,d}$$

means that $p - \sum_{0\leqq j<J} p_j \in S^{m_J,d}$ for all J. In particular, $p - p_0 \in S^{m_1,d} \subset S^{m_0,d}$, so $p \in S^{m_0,d}$; and we also say that $p \sim \sum_{j\in\mathbb{N}} p_j$ in $S^{m_0,d}$.

The symbol classes $S^{m,d}$ have the usual property that there is a symbol associated with any given asymptotic expansion (proved by standard methods, taking the uniformity in z into account):

Lemma 1.9. *When a sequence p_j is given as in Definition 1.8, then there exists a $p \in S^{m_0,d}$ such that $p \sim \sum_{j\in\mathbb{N}} p$ in $S^{\infty,d}$.*

Polyhomogeneous symbols in the calculus are defined as follows:

Definition 1.10. *$p \in S^{\infty,d}(\mathbb{R}^\nu, \mathbb{R}^n, \Gamma)$ is said to be* **(weakly) polyhomogeneous** *(wphg) if there exists a sequence of symbols $p_j \in S^{m_j-d,d}$, homogeneous in (ξ,μ) for $|\xi| \geqq 1$ of degree $m_j \searrow -\infty$, such that $p \sim \sum_{j=0}^\infty p_j$ in $S^{\infty,d}$. In this case, we say that $p \in S^{\infty,d}_{\mathrm{wphg}}(\mathbb{R}^\nu, \mathbb{R}^n, \Gamma)$ (or $p \sim \sum_{j\in\mathbb{N}} p_j$ in $S^{\infty,0}_{\mathrm{wphg}}(\mathbb{R}^\nu, \mathbb{R}^n, \Gamma)$), with degrees $\{m_j\}_{j\in\mathbb{N}}$ and μ-exponent d. In particular, $p \in S^{m_0-d,d}$.*

Note that homogeneity is not required of the p_j in the set $\{(\xi,\mu) \mid |\xi| \leqq 1\}$. We shall also consider a more restricted class with homogeneity for $|\xi,\mu| \geqq 1$ (strongly polyhomogeneous symbols), so the word "weakly" is used when we want to distinguish the general type from the special type.

Example 1.11. When a classical polyhomogeneous symbol $p(x,\xi)$ of degree m is considered as a function of (x,ξ,μ), constant in $\mu \in \mathbb{C}$, it belongs to $S^{\infty,0}_{\mathrm{wphg}}(\mathbb{R}^\nu, \mathbb{R}^N, \Gamma)$ with degrees $\{m-j\}_{j\in\mathbb{N}}$ and μ-exponent 0. In particular, $p \in S^{m,0}$.

We shall now show that our symbols have a *second* asymptotic expansion, regardless of whether they are polyhomogeneous or not. Our proof of asymptotic trace estimates in Section 2 will use *both* types of expansion.

Theorem 1.12. *For* $p \in S^{m,d}(\mathbb{R}^\nu, \mathbb{R}^n, \Gamma)$, *set*

$$p_{(d,k)}(x,\xi) = \tfrac{1}{k!}\partial_z^k(z^d p(x,\xi,\tfrac{1}{z}))|_{z=0}. \tag{1.6}$$

Then $p_{(d,k)} \in S^{m+k}(\mathbb{R}^\nu, \mathbb{R}^n)$, *and for any* N,

$$p(x,\xi,\mu) - \textstyle\sum_{0\leqq k<N}\mu^{d-k}p_{(d,k)}(x,\xi) \in S^{m+N,d-N}(\mathbb{R}^\nu, \mathbb{R}^n, \Gamma). \tag{1.7}$$

Proof. In view of (1.4 iii), we can take $d=0$. As usual we ignore the dependence on x. Let Γ' be a closed subsector of Γ. By assumption (cf. Definition 1.1), the k'th z-derivative of $p(\xi,\frac{1}{z})$ is bounded in S^{m+k} for $|z| \leqq 1$, z in the sector $\Gamma'' = \{z \mid \frac{1}{z} \in \Gamma'\}$. Hence p and its derivatives have limits for $z \to 0$ in Γ'', e.g., if $z \in \mathbb{R}_+ \subset \Gamma''$,

$$p_{(0,0)}(\xi) = -\lim_{z\to 0, z\in\mathbb{R}_+}\int_z^1 (\partial_z p)(\xi,\tfrac{1}{r})dr + p(\xi,1), \text{ in } S^{m+1}.$$

Similarly for $k \geqq 0$, the symbols $p_{(0,k)}(\xi)$ are well-defined as limits of $\partial_z^k p(\xi,\frac{1}{z})$ in S^{m+k+1}. Moreover, the uniform S^{m+k} estimates of $\partial_z^k p(\xi,\frac{1}{z})$ imply that $p_{(0,k)}(\xi) \in S^{m+k}$. The $p_{(0,k)}(\xi)$ are in fact the Taylor coefficients of $p(\xi,\frac{1}{z})$ at $z=0$, and (1.7) is a Taylor expansion (note that $\mu^{-k} = z^k$). By Taylor's formula,

$$\mu^N\Big[p(\xi,\mu) - \sum_{k<N}\mu^{-k}p_{(0,k)}(\xi)\Big] = z^{-N}\Big[p(\xi,\tfrac{1}{z}) - \sum_{k<N}z^k p_{(0,k)}(\xi)\Big]$$
$$= \tfrac{1}{N!}\int_0^1 (1-t)^{N-1}(\partial_z^N p)(\xi,\tfrac{1}{tz})dt,$$

and we want to show that this lies in $S^{m+N,0}$ as a function of all the variables. That will imply (1.7).

Let z belong to a ray $z = e^{i\theta}r$; assume for simplicity that $\theta = 0$. By assumption, $z \mapsto \partial_z^N p(\cdot,\frac{1}{z})$ is bounded in S^{m+N} for $z \leqq 1$. Then for any $k \in \mathbb{N}$,

$$\partial_z^k\int_0^1(1-t)^{N-1}(\partial_z^N p)(\xi,\tfrac{1}{tz})dt = \int_0^1 t^k(1-t)^{N-1}(\partial_z^{N+k}p)(\xi,\tfrac{1}{tz})dt$$

is bounded in S^{m+N+k}. Since this holds locally uniformly in θ, and the functions are holomorphic in z, we obtain (1.7). □

Observe the particular feature of (1.7) that the remainders in the expansion lie in spaces that with increasing N grow successively *better* in μ-decay (for $|\mu| \to \infty$), but *worse* in ξ-decay (for $|\xi| \to \infty$). This same effect is seen more directly in the geometric expansion of the simple example

$$p(x,\xi,\mu) = \frac{1}{|\xi|^2+\mu^2} = \mu^{-2}\Big(1 - \frac{|\xi|^2}{\mu^2} + \frac{|\xi|^4}{\mu^4} - \dots\Big).$$

In this respect, property (1.7) is somewhat reminiscent of the transmission condition of Boutet de Monvel [BM]; but it differs by requiring the asymptotic property for $\mu \to \infty$ only on rays (halflines) whereas the matching for $\xi_n \to +\infty$ and $\xi_n \to -\infty$ is important in the transmission condition. The property is also related to semi-classical expansions. (One could study symbol classes having a more general expansion with terms $z^d(\log z)^j$, at the cost of more elaborate explanations; but the present class is adequate for our immediate purposes, and composition rules here follow easily from the standard rules in view of the smoothness for $z \to 0$.)

We shall now consider some important special cases.

Lemma 1.13. *Let $m \in \mathbb{Z}$. Denote as usual $\frac{1}{\mu}$ by z.*

For $m \leqq 0$, $[\xi,\mu]^m$ and $|\xi,\mu|^m$ are in $S^{m,0}(\mathbb{R}^\nu,\mathbb{R}^n,\mathbb{R}_+) \cap S^{0,m}(\mathbb{R}^\nu,\mathbb{R}^n,\mathbb{R}_+)$, and $\langle z\xi\rangle^m \in S^{m,-m}(\mathbb{R}^\nu,\mathbb{R}^n,\mathbb{R}_+) \cap S^{0,0}(\mathbb{R}^\nu,\mathbb{R}^n,\mathbb{R}_+)$.

For $m \geqq 0$, $[\xi,\mu]^m$ and $|\xi,\mu|^m$ are in $S^{m,0}(\mathbb{R}^\nu,\mathbb{R}^n,\mathbb{R}_+) + S^{0,m}(\mathbb{R}^\nu,\mathbb{R}^n,\mathbb{R}_+)$ and $\langle z\xi\rangle^m \in S^{m,-m}(\mathbb{R}^\nu,\mathbb{R}^n,\mathbb{R}_+) + S^{0,0}(\mathbb{R}^\nu,\mathbb{R}^n,\mathbb{R}_+)$.

Proof. Let $p(\xi,\mu) = [\xi,\mu]^m$. It follows from the homogeneity that

$$\partial_\xi^\alpha|\xi,\mu|^m = O(|\xi,\mu|^{m-|\alpha|}), \text{ for all } \alpha,$$

so p and $|\xi,\mu|^m$ are in S^m for each fixed $\mu > 0$; p is so for $\mu = 0$ also. We have to show uniformity of estimates for $\mu \geqq 1$ (i.e. $z \leqq 1$); here

$$p(\xi,\mu) = |\xi,\mu|^m = z^{-m}(|z\xi|^2+1)^{m/2} = z^{-m}\langle z\xi\rangle^m. \tag{1.8}$$

The case $m \leqq 0$. Since $z^m p(\xi,\frac{1}{z}) = f(\xi/\mu)$ with $f(\eta) = \langle\eta\rangle^m \in S^m \subset S^0$, it follows immediately from Lemma 1.3 that $z^m p \in S^{0,0}$ and hence $p \in S^{0,m}$. Now let us show that $p \in S^{m,0}$. When $|\xi| \leqq 1$ and $0 < z \leqq 1$, all derivatives of (1.8) are bounded (since $-m \in \mathbb{N}$). Next, let $|\xi| \geqq 1$. Divide $\mathbb{R}^n$ into sectors where $|\xi_k| \geqq |\xi|/2\sqrt{n}$ for some $k = 1,\dots,n$. In such a sector,

$$p(\xi,\mu) = \mu^m p(\xi/\mu,1) = \xi_k^m[(\xi_k/\mu)^{|m|}p(\xi/\mu,1)] =: \xi_k^m f_k(\xi/\mu)$$

with $f_k(\eta) = \eta_k^{|m|}p(\eta,1) \in S^0(\mathbb{R}^n)$. Moreover, ξ_k^m can be extended from $\{\xi \in \mathbb{R}^n \mid 2\sqrt{n}|\xi_k| \geqq |\xi| \geqq 1\}$ to the full space, to give a symbol $q_k(\xi) \in S^m$. By Lemma 1.3, $f_k(\xi/\mu) \in S^{0,0}$, so after multiplication by $q_k(\xi)$ we obtain $p \in S^{m,0}$. Then by (1.8), $\langle z\xi\rangle^m \in S^{m,-m}$, as claimed in the lemma.

The case $m > 0$. We divide $\mathbb{R}^n$ into sectors as above, and write

$$|\xi,\mu|^m = (\xi_k^m + \mu^m)(\xi_k^m + \mu^m)^{-1}|\xi,\mu|^m =: (\xi_k^m + \mu^m)q(\xi,\mu).$$

Then q is of the form $f(\xi/\mu)$ with $f(\eta) = (\eta_k^m + 1)^{-1}\langle\eta\rangle^m \in S^0$ in the appropriate sector, and Lemma 1.3 implies that $q \in S^{0,0}$ there. Clearly,

$$\xi_k^m + \mu^m \in S^{m,0} + S^{0,m},$$

cf. Example 1.1. Then the product rule, Lemma 1.6, gives:

$$|\xi,\mu|^m \in (S^{m,0} + S^{0,m})S^{0,0} \subset S^{m,0} + S^{0,m}. \qquad \square$$

This is used in the following lemma.

Lemma 1.14. *Let $p(x,\xi,\mu)$ be C^∞ in $(x,\xi,\mu)\in \mathbb{R}^\nu\times\mathbb{R}^n\times(\Gamma\cup\{0\})$, such that for $|\xi,\mu|\geqq 1$, p is homogeneous in (ξ,μ) of degree $m\in\mathbb{Z}$ and holomorphic in $\mu\in\overset{\circ}{\Gamma}$.*

If $m\leqq 0$, then $p\in S^{m,0}(\mathbb{R}^\nu,\mathbb{R}^n,\Gamma)\cap S^{0,m}(\mathbb{R}^\nu,\mathbb{R}^n,\Gamma)$.

For general m, $p\in S^{m,0}+S^{0,m}$, and $\partial_x^\beta\partial_\xi^\alpha p\in S^{m-|\alpha|,0}\cap S^{0,m-|\alpha|}$ when $|\alpha|\geqq m$.

Proof. If $m=0$, we have on each ray $\mu=\varrho e^{i\theta}$, $\varrho>0$, a symbol of the type considered in Example 1.4, where it was shown to satisfy the estimates for $S^{0,0}$. The estimates are locally uniform in θ, so the lemma follows, for the case $m=0$, when we moreover note that the estimates of the radial derivatives give estimates of the z-derivatives in view of the analyticity.

When m is general, we write (for each ray $\mu=\varrho e^{i\theta}$),

$$p(x,\xi,\varrho e^{i\theta})=[\xi,\varrho]^m\big([\xi,\varrho]^{-m}p(x,\xi,\varrho e^{i\theta})\big).$$

Now if $m\leqq 0$, $[\xi,\varrho]^m\in S^{m,0}\cap S^{0,m}$ by Lemma 1.5, and $[\xi,\varrho]^{-m}p(x,\xi,\varrho e^{i\theta})$ is of the preceding type, homogeneous of degree 0 in (ξ,ϱ) for $|\xi,\varrho|\geqq 1$; hence lies in $S^{0,0}$ by Example 1.4. The product rule (Lemma 1.6) then gives that $p\in S^{m,0}\cap S^{0,m}$ as a function of (ξ,ϱ); and the statement extends to the function of (ξ,μ) in view of the analyticity and local uniformity in θ.

If $m\geqq 0$, we use the fact that $[\xi,\mu]^m\in S^{m,0}+S^{0,m}$ in a similar way.

Since $\partial_x^\beta\partial_\xi^\alpha p$ satisfies the hypotheses with m replaced by $m-|\alpha|$, the last statement in the lemma follows from the case $m\leqq 0$. □

Note that smoothness (plus homogeneity) merely for $(\xi,\mu)\in\mathbb{R}^n\times\Gamma$ would not suffice. E.g. $\mu^{-1}(|\xi|^2+\mu^2)^{\frac12}$ is smooth and homogeneous of degree 0 for $(\xi,\mu)\in\mathbb{R}^n\times\mathbb{R}_+$; but it is not smooth at $\mu=0$, $|\xi|\geqq 1$, and not uniformly in S^0 for $\mu\geqq 1$.

Symbols satisfying the hypotheses in Lemma 1.14 will be called strongly homogeneous. More generally, we define:

Definition 1.15. *A symbol $p(x,\xi,\mu)$ is said to be* ***strongly polyhomogeneous*** *(sphg) of degree $m\in\mathbb{R}$ with respect to $(\xi,\mu)\in\mathbb{R}^n\times(\Gamma\cup\{0\})$, when $p(x,\xi,\mu)\in C^\infty(\mathbb{R}^\nu\times\mathbb{R}^n\times(\Gamma\cup\{0\}))$ and there is a sequence of functions $p_j\in C^\infty(\mathbb{R}^\nu\times\mathbb{R}^n\times(\Gamma\cup\{0\}))$ that for $|\xi,\mu|\geqq 1$ are homogeneous in $(\xi,\mu)\in\mathbb{R}^n\times\Gamma$ of degree $m-j$, with p and the p_j holomorphic in $\mu\in\overset{\circ}{\Gamma}$ and*

$$\partial_x^\beta\partial_\xi^\alpha\partial_\mu^k\Big(p-\sum_{0\leqq j<J}p_j\Big)=O([\xi,\mu]^{m-J-|\alpha|-k})\ \text{for}\ |\xi,\mu|\geqq 1,\tag{1.9}$$

for all indices, uniformly for μ in closed subsectors of $\Gamma\cup\{0\}$. (One can replace $|\xi,\mu|\geqq 1$ by $|\xi,\mu|\geqq\varepsilon_j$, with positive ε_j.)

This class was studied earlier in Shubin [Sh]. Symbols $p(x,\xi,\mu)=q(x,(\xi,\mu))$, where $q(x,\zeta)\in S^m(\mathbb{R}^\nu,\ \mathbb{R}^{n+1})$ is classical polyhomogeneous with $m\in\mathbb{Z}$, satisfy Definition 1.15 with $\Gamma\cup\{0\}=\mathbb{R}$.

Theorem 1.16. *Let $m \in \mathbb{Z}$. Let $p(x,\xi,\mu)$ be strongly polyhomogeneous of degree m in $(\xi,\mu) \in \mathbb{R}^n \times (\Gamma \cup \{0\})$. Then $p \in S^{m,0}(\mathbb{R}^\nu, \mathbb{R}^n, \Gamma) + S^{0,m}(\mathbb{R}^\nu, \mathbb{R}^n, \Gamma)$, and*

$$\partial_x^\beta \partial_\xi^\alpha p \in S^{m-|\alpha|,0}(\mathbb{R}^\nu, \mathbb{R}^n, \Gamma) \cap S^{0,m-|\alpha|}(\mathbb{R}^\nu, \mathbb{R}^n, \Gamma), \quad \text{for } |\alpha| - m \geqq 0, \text{ all } \beta.$$

In particular, $p \in S^{m,0}(\mathbb{R}^\nu, \mathbb{R}^n, \Gamma) \cap S^{0,m}(\mathbb{R}^\nu, \mathbb{R}^n, \Gamma)$ if $m \leqq 0$.

As a consequence, classical polyhomogeneous symbols in $n+1$ cotangent variables give strongly polyhomogeneous symbols in n cotangent variables, when one cotangent variable is replaced by μ (here $\Gamma = \mathbb{R}_+ \cup \mathbb{R}_-$).

Proof. Note that $\partial_x^\beta \partial_\xi^\alpha p$ satisfies the hypotheses with m replaced by $m - |\alpha|$. By Lemma 1.14, the terms p_j in p with $j \leqq m$ are in $S^{m-j,0} + S^{0,m-j}$ and those with $j \geqq m$ are in $S^{m-j,0} \cap S^{0,m-j}$. Then it remains to show that $r_J = p - \sum_{0 \leqq j < J} p_j \in S^{m-J,0} \cap S^{0,m-J}$ for large $J \geqq m$, i.e. that

$$\text{(1.10)} \qquad \begin{aligned} |\partial_x^\beta \partial_\xi^\alpha \partial_z^k r_J(x,\xi,\tfrac{1}{z})| &\leqq C[\xi]^{m-J-|\alpha|+k}, \\ |\partial_x^\beta \partial_\xi^\alpha \partial_z^k z^{m-J} r_J(x,\xi,\tfrac{1}{z})| &\leqq C[\xi]^{-|\alpha|+k}, \end{aligned}$$

for all α, β, k, when $0 < |z| \leqq 1$, $\frac{1}{z}$ in a closed subsector of Γ. By hypothesis,

$$\partial_x^\beta \partial_\xi^\alpha \partial_\mu^k r_J(x,\xi,\mu) = O([\xi,\mu]^{m-J-|\alpha|-k}) \text{ for all } \alpha, \beta, k.$$

Since

$$\partial_z r_J(x,\xi,\tfrac{1}{z}) = -\mu^2 \partial_\mu r_J(x,\xi,\mu), \text{ with } \mu = \tfrac{1}{z},$$

$$[\xi,\mu]^a \leqq C[\xi]^a, \quad |z|^a [\xi,\mu]^a \leqq C, \text{ when } a \leqq 0, \ |z| \leqq 1,$$

we find for $|z| \leqq 1$, $l \in \mathbb{N}$,

$$\begin{aligned} |\partial_x^\beta \partial_\xi^\alpha \partial_z^k (z^{-l} r_J(x,\xi,\tfrac{1}{z}))| &\leqq C \sum_{0 \leqq k' \leqq k} |\mu|^{k+l+k'} |\partial_x^\beta \partial_\xi^\alpha \partial_\mu^{k'} r_J| \\ &\leqq C' \sum_{0 \leqq k' \leqq k} |z|^{-k-l-k'} [\xi,\mu]^{m-J-|\alpha|-k'} \\ &\leqq C'' [\xi]^{m-J-|\alpha|+k+l}. \end{aligned}$$

For $l = 0$ resp. $J - m$, this shows the desired estimates. □

When $a(x,\xi)$ is the principal symbol of a strongly elliptic *differential operator* A, of order 2, say, then $p = (a(x,\xi) + \mu^2)^{m/2}$ is, for any $m \in \mathbb{Z}$, homogeneous of degree m and C^∞ in $(\xi,\mu) \in (\mathbb{R}^n \times (\Gamma \cup \{0\})) \setminus \{0,0\}$, where $\Gamma = \{\mu \in \mathbb{C} \setminus \{0\} \mid |\arg \mu| < \pi/4 + \delta\}$ for some $\delta > 0$. It is holomorphic in $\mu \in \Gamma$, and it can be modified for $|\xi,\mu| \leqq 1$ to be C^∞ also at $(0,0)$, e.g. by multiplication by an excision function $\theta_1(|\xi,\mu|)$, where

$$\text{(1.11)} \quad \theta_1(t) \in C^\infty(\mathbb{R}), \quad \theta_1(t) = 1 \text{ for } |t| \geqq 1, \quad \theta_1(t) = 0 \text{ for } |t| \leqq \tfrac{1}{2}.$$

Then we get a symbol as in the above theorem (in fact as in Lemma 1.14). For example,

$$(a(x,\xi)+\mu^2)^{-\frac{1}{2}}\theta_1(|\xi,\mu|)\in S^{-1,0}\cap S^{0,-1}. \tag{1.12}$$

It also follows that the full parametrix symbol for the resolvent $(A+\mu^2)^{-1}$ is as in the theorem with $m=-2$. These symbols are of *regularity* $+\infty$ in the terminology of the book Grubb [G1].

In contrast, if $a(x,\xi)$ is truly pseudodifferential, e.g. $a=[\xi]$, and we consider $a(x,\xi)+\mu$, the fall-off for $\mu\to\infty$ of the higher derivatives exhibited in (1.9) (with $J=0$) does not hold; e.g. we have only

$$\partial_\xi^\alpha([\xi]+\mu)=O([\xi]^{1-|\alpha|}),\quad \partial_\xi^\alpha([\xi]+\mu)^{-1}=O([\xi]^{1-|\alpha|}[\xi,\mu]^{-2}),\ \text{for } |\alpha|>0.$$

This difficulty is handled in [G1] by the introduction of symbol classes of finite *regularity* $\nu\in\mathbb{R}$. We here take a different point of view, for the purpose of keeping track of full asymptotic expansions in μ.

Theorem 1.17. *Let $a(x,\xi)$ be homogeneous of integer degree $m>0$ in ξ for $|\xi|\geqq 1$ and C^∞ in $(x,\xi)\in\mathbb{R}^\nu\times\mathbb{R}^n$; and assume that $a(x,\xi)+\mu^m$ is invertible for $(x,\xi,\mu)\in\mathbb{R}^\nu\times\mathbb{R}^n\times(\Gamma\cup\{0\})$. Set $p(x,\xi,\mu)=(a(x,\xi)+\mu^m)^{-1}$. Then $p\in S^{0,-m}(\mathbb{R}^\nu,\mathbb{R}^n,\Gamma)\cap S^{-m,0}(\mathbb{R}^\nu,\mathbb{R}^n,\Gamma)$.*

Proof. As observed above, $(a+\mu^m)^{-1}$ is generally not strongly homogeneous as in Lemma 1.14, so another argument must be given.

To show that $p\in S^{0,-m}$, consider $f(x,\xi,r)=\mu^m p=z^{-m}p=(z^m a+1)^{-1}$ $=(r^m e^{im\theta}a+1)^{-1}$, with $z=re^{i\theta}$, $0<r\leqq 1$. As usual, we ignore x; and we can take $\theta=0$ for simplicity. We must show that $f\in S^{0,0}$. We have

$$f=O(\langle r\xi\rangle^{-m}). \tag{1.13}$$

This is clear when $|\xi|\leqq 1$, and when $|\xi|\geqq 1$,

$$|r^m a(\xi)+1|=|a^h(r\xi)+1|\geqq C\langle r\xi\rangle^m,$$

where a^h is the homogeneous extension of a from $\{|\xi|\geqq 1\}$. This is checked separately for $r\xi$ small and for $r\xi$ large. Next, we observe that if $b\in S^j$, $j=1,2,\ldots$, then

$$\partial_\xi^\alpha(r^j b)=O(\langle\xi\rangle^{-|\alpha|}\langle r\xi\rangle^j). \tag{1.14}$$

For, $r^j\partial_\xi^\alpha b=\langle\xi\rangle^{-|\alpha|}(r^j\langle\xi\rangle^{|\alpha|}\partial_\xi^\alpha b)=\langle\xi\rangle^{-|\alpha|}r^jO(\langle\xi\rangle^j)=O(\langle\xi\rangle^{-|\alpha|}\langle r\xi\rangle^j)$. Now, $\partial_\xi f=-\langle\xi\rangle^{-1}r^m[\langle\xi\rangle\partial_\xi a]f^2$, and generally,

$$\partial_\xi^\alpha f=\langle\xi\rangle^{-|\alpha|}\sum_{1\leqq j\leqq|\alpha|}r^{mj}b_{\alpha,j}f^{j+1}\quad\text{with } b_{\alpha,j}\in S^{mj}.$$

Together, (1.13) and (1.14) give

$$\partial_\xi^\alpha f=\langle\xi\rangle^{-|\alpha|}O(\langle r\xi\rangle^{-m}). \tag{1.15}$$

Finally, $\partial_r f = -\langle\xi\rangle[mr^{m-1}\langle\xi\rangle^{-1}af^2] = \langle\xi\rangle[r^{m-1}b_{m-1}f^2]$, with $b_{m-1} \in S^{m-1}$, and generally

$$(1.16) \qquad \partial_r^j f = \langle\xi\rangle^j \textstyle\sum_{j \leqq km \leqq jm} r^{km-j} b_{k,j} f^{k+1}, \quad \text{with } b_{k,j} \in S^{km-j}.$$

By (1.15) and (1.16), $\langle\xi\rangle^{|\beta|-j}\partial_\xi^\beta \partial_r^j f = O(\langle r\xi\rangle^{-j-m})$. This implies that $\partial_r^j f \in S^{m+j}$, uniformly in r, for all j, so $f \in S^{0,0}$ and hence $p \in S^{0,-m}$.

To see that $p \in S^{-m,0}$, we note that

$$ap = \frac{a}{a+\mu^m} = 1 - \frac{\mu^m}{a+\mu^m} = 1 - f \in S^{0,0}.$$

Now since $a(x,\xi)^{-1}$ exists and is homogeneous of degree $-m$ for $|\xi| \geqq 1$, a^{-1} belongs to $S^{-m} \subset S^{-m,0}$ (cf. Example 1.2). Then $p = a^{-1}ap \in S^{-m,0} \cdot S^{0,0} \subset S^{-m,0}$, by Lemma 1.6. □

The two cases treated in Theorems 1.16 and 1.17 are fundamental for the symbol types we need to treat, which often arise from these by taking compositions and inverses etc.

1.2. Rules of calculus

When $p(x,y,\xi,\mu) \in S^{m,d}(\mathbb{R}^{2n}, \mathbb{R}^n, \Gamma)$, one defines the ψdo $P = \mathrm{OP}(p)$ (which depends on the parameter μ) by

$$(1.17) \qquad \mathrm{OP}(p)f(x) = \int e^{i(x-y)\cdot\xi} p(x,y,\xi,\mu) f(y)\,dy\,đ\xi,$$

with $đ\xi = (2\pi)^{-n}d\xi$. The operator is defined for general functions and distributions under suitable hypotheses on the behavior of p in x, y and suitable interpretations; see e.g. [S4], [H2,3], [T], [Sh]. We think of $\mathbb{R}^{2n}$ as $\mathbb{R}^n \times \mathbb{R}^n$ and use x as the variable in the first copy, y as the variable in the second.

The rules of calculus for pseudodifferential operators, shown in each of the mentioned works for slightly different symbol classes, extend to our parameter-dependent symbol spaces in a straightforward way, since we are dealing with spaces of symbols with estimates uniform in the parameter z (for $|z| \leqq 1$). Let us just state the general result, leaving the choice of the symbol class to the applications.

Theorem 1.18. 1° *Let* $p(x,y,\xi,\mu) \in S^{m,d}(\mathbb{R}^{2n}, \mathbb{R}^n, \Gamma)$. *Then*

$$(1.18) \qquad \mathrm{OP}(p(x,y,\xi,\mu)) = \mathrm{OP}(q(x,\xi,\mu)) = \mathrm{OP}(q_1(y,\xi,\mu)),$$

where q *and* q_1 *are is in* $S^{m,d}(\mathbb{R}^n, \mathbb{R}^n, \Gamma)$, *and*

$$q(x,\xi,\mu) \sim \sum_{\alpha\in\mathbb{N}^n} \tfrac{1}{\alpha!}(-i\partial_y)^\alpha \partial_\xi^\alpha p(x,x,\xi,\mu),$$

$$(1.19) \qquad q_1(y,\xi,\mu) \sim \sum_{\alpha\in\mathbb{N}^n} \tfrac{1}{\alpha!}(i\partial_x)^\alpha \partial_\xi^\alpha p(y,y,\xi,\mu), \quad \textit{in } S^{m,d}.$$

2° *If* $p(x,\xi,\mu) \in S^{m,d}(\mathbb{R}^n,\mathbb{R}^n,\Gamma)$, *then*

$$\operatorname{OP}(p(x,\xi,\mu))^* = \operatorname{OP}(p^*(y,\xi,\mu)) = \operatorname{OP}(p_1(x,\xi,\mu)), \text{ where}$$

$$p_1(x,\xi,\mu) \sim \sum_{\alpha\in\mathbb{N}^n} \tfrac{1}{\alpha!}(-i\partial_x)^\alpha \partial_\xi^\alpha p^*(x,\xi,\mu) \text{ in } S^{m,d}. \tag{1.20}$$

3° *Let* $p(x,\xi,\mu) \in S^{m,d}(\mathbb{R}^n,\mathbb{R}^n,\Gamma)$ *and* $p'(x,\xi,\mu) \in S^{m',d'}(\mathbb{R}^n,\mathbb{R}^n,\Gamma)$. *Then*

$$\operatorname{OP}(p)\operatorname{OP}(p') = \operatorname{OP}(q), \text{ where}$$

$$q(x,\xi,\mu) \sim \sum_{\alpha\in\mathbb{N}^n} \tfrac{1}{\alpha!}\partial_\xi^\alpha p(x,\xi,\mu)(-i\partial_x)^\alpha p'(x,\xi,\mu) \text{ in } S^{m+m',d+d'}. \tag{1.21}$$

In each of these rules, the resulting symbol is polyhomogeneous resp. strongly polyhomogeneous if the initial symbols are so.

The stated identities for the operators hold without restriction for the symbol classes in [S4] with rapid decrease in x, and for those in [H3, 18.1] with global boundedness in x. For the symbol classes in [H2], [T], [Sh] with local boundedness in x, they hold under suitable additional assumptions on rapid decrease or compact support (and can be stated more generally as equivalences modulo smoothing operators). In the formula for the composition of two symbols one uses the Leibniz formula as in Lemma 1.6.

Strong polyhomogeneity is preserved under these rules, since the formulas in the strongly polyhomogeneous case agree with the formulas for the corresponding manipulations with symbols $p(x,x_{n+1},\xi,\xi_{n+1})$ (resp. $p(x,x_{n+1},y,y_{n+1},\xi,\xi_{n+1})$) on $\mathbb{R}^{n+1}$, when these are constant in x_{n+1} (resp. (x_{n+1},y_{n+1})), and μ is identified with ξ_{n+1}.

Definition 1.19. *For p and p' as in Theorem 1.18* 3°, *the resulting symbol q is denoted $p \circ p'$.*

Remark 1.20. With a slight abuse of notation we shall speak of operators as being in x-form, in y-form, or in (x,y)-form, when they are written as in (1.17) with a symbol $p(x,\xi,\mu)$, $p(y,\xi,\mu)$ resp. $p(x,y,\xi,\mu)$; in the proof of the above theorem one uses the passage from one form to another. This can also be done with respect to part of the coordinates only; for example (1.17) with a symbol $p(x',y_n,\xi,\mu)$ (where $x' = (x_1,\dots,x_{n-1})$) is said to be "in (x',y_n)-form." This is useful in the consideration of boundary operators, cf. Lemma A.1.

Proposition 1.21. *When $q \in S^{-\infty,d}(\mathbb{R}^{2n},\mathbb{R}^n,\Gamma)$, the Schwartz kernel $K(x,y,\mu)$ of* $\operatorname{OP}(q)$ *has an expansion*

$$K(x,y,\mu) \sim \sum_{k=0}^{\infty} K_k(x,y)\mu^{d-k}, \tag{1.22}$$

where K_k is in $C^\infty(\mathbb{R}^{2n})$, and $K - \sum_{k<N} K_k\mu^{d-k}$ is in $C^\infty(\mathbb{R}^{2n} \times \Gamma)$, holomorphic in $\mu \in \mathring{\Gamma}$ for $|\mu| \geqq 1$ and $O(\mu^{d-N})$ in closed subsectors, for all N.

This holds in particular for $\varphi\operatorname{OP}(p)\psi = \operatorname{OP}\big(\varphi(x)p(x,y,\xi,\mu)\psi(y)\big)$, *when $p \in S^{m,d}(\mathbb{R}^{2n},\mathbb{R}^n,\Gamma)$ and $\varphi,\psi \in C^\infty(\mathbb{R}^n)$ with $\varphi = 0$ on the support of ψ.*

Proof. By Theorem 1.12, there are symbols $q_{(d,k)} \in S^{-\infty}(\mathbb{R}^{2n}, \mathbb{R}^n)$ such that

$$q(x,y,\xi,\mu) - \sum_{k<N} \mu^{d-k} q_{(d,k)}(x,y,\xi) \in S^{-\infty,d-N}, \text{ for all } N.$$

Then

$$K(x,y,\mu) = \int e^{i(x-y)\cdot\xi} q(x,y,\xi,\mu)\, đ\xi, \quad K_k(x,y) = \int e^{i(x-y)\cdot\xi} q_{(d,k)}(x,y,\xi)\, đ\xi,$$

are the desired kernels.

For $\varphi \mathrm{OP}(p)\psi = \mathrm{OP}(\varphi(x)p(x,y,\xi,\mu)\psi(y))$, we have by Theorem 1.18 that it equals $\mathrm{OP}(q(x,\xi,\mu))$, where

$$q(x,\xi,\mu) \sim \sum_{\alpha\in\mathbb{N}^n} \tfrac{1}{\alpha!}(-i\partial_y)^\alpha \partial_\xi^\alpha \big(\varphi(x)p(x,y,\xi,\mu)\psi(y)\big)\big|_{y=x} \text{ in } S^{m,d}.$$

When $\varphi = 0$ on $\operatorname{supp}\psi$, each term in the series is zero, so $q \in S^{-\infty,d}$. □

The operator classes are invariant under coordinate changes (again the proof is like the usual one, when one takes the uniform estimates in $|z| \leqq 1$ into account):

Proposition 1.22. *Let $\chi : U \to V$ be a diffeomorphism between U and V, open subsets of $\mathbb{R}^n$. Let $p(x,y,\xi,\mu) \in S^{m,d}(V \times V, \mathbb{R}^n, \Gamma)$ and suppose that p vanishes for (x,y) outside a compact subset of $V \times V$. Set*

$$(1.23) \qquad Qf(x) = \mathrm{OP}(p)(f \circ \chi^{-1})(\chi(x)), \quad x \in U.$$

Then $Q = \mathrm{OP}(q)$ with $q \in S^{m,d}(U \times U, \mathbb{R}^n, \Gamma)$.

In fact, if $\Phi = (\Phi_{jk})_{j,k\leqq n}$ satisfies $\langle \chi(x) - \chi(y), \xi\rangle = \sum_{k,j\leqq n} \Phi_{kj}(x,y)(x_k - y_k)\xi_j$ (so that $\Phi(x,x) = \partial\chi/\partial x$, and $\Phi(x,y)$ is invertible for y near x), and we reduce (by Proposition 1.21) to the case where $p(x,y,\xi,\mu)$ vanishes on the set of (x,y) where Φ is not invertible, then $Q = \mathrm{OP}(q')$ with

$$(1.24) \qquad q'(x,y,\xi,\mu) = p(\chi(x), \chi(y), {}^t\Phi^{-1}(x,y)\xi, \mu) J(x,y),$$

where J is a Jacobian factor.

An interesting question is, of course, how to invert operators in this calculus. We shall not give a general discussion or formal definition of ellipticity, but just show an important special case, useful in Section 3.

Theorem 1.23. *Let $p(x,\xi,\mu) \in S^{0,0}(\mathbb{R}^\nu, \mathbb{R}^n, \Gamma)$ be such that $p = p_0 + p_1$ with $p_1 \in S^{-1,0}$ and p_0^{-1} in C^∞ and uniformly bounded in (ξ,μ), locally uniformly in x and uniformly for μ in closed subsectors of Γ, $|\mu| \geqq 1$.*

For any bounded open set U there exists a symbol $q_U(x,\xi,\mu) \in S^{0,0}(\mathbb{R}^\nu, \mathbb{R}^n, \Gamma)$ such that $p \circ q_U \sim 1$ in $S^{0,0}$ on U; here q_U has an asymptotic expansion as described below in (1.26).

If p is polyhomogeneous resp. strongly polyhomogeneous, so are q_U and the remainder $p \circ q_U - 1$.

Proof. This is modeled on the standard proof. Let $q_0 = p_0^{-1}$. By the Leibniz formula,

$$(1.25) \qquad 0 = \partial_\xi^\alpha(p_0 q_0) = \sum_{\beta \leqq \alpha} \binom{\alpha}{\beta} \partial_\xi^{\alpha-\beta} p_0 \partial_\xi^\beta q_0,$$

$$\text{hence } \partial_\xi^\alpha q_0 = -q_0 \sum_{\beta < \alpha} \binom{\alpha}{\beta} \partial_\xi^{\alpha-\beta} p_0 \, \partial_\xi^\beta q_0;$$

$$0 = \partial_z^j(p_0 q_0) = \sum_{k \leqq j} \binom{j}{k} \partial_z^{j-k} p_0 \partial_z^k q_0,$$

$$\text{hence } \partial_z^j q_0 = -q_0 \sum_{k < j} \binom{j}{k} \partial_z^{j-k} p_0 \, \partial_z^k q_0.$$

Let us show that $q_0 \in S^{0,0}$. From the first formula we get successively:

$$\partial_{\xi_1} q_0 = -q_0(\partial_{\xi_1} p_0) q_0 = O(1) O(\langle\xi\rangle^{-1}) O(1) = O(\langle\xi\rangle^{-1}),$$

$$\vdots$$

$$\partial_\xi^\alpha q_0 = -q_0 \sum_{\beta < \alpha} \binom{\alpha}{\beta} (\partial_\xi^{\alpha-\beta} p_0)(\partial_\xi^\beta q_0)$$

$$= O(1) \sum_{\beta < \alpha} \binom{\alpha}{\beta} O(\langle\xi\rangle^{-|\alpha-\beta|}) O(\langle\xi\rangle^{-|\beta|}) = O(\langle\xi\rangle^{-|\alpha|}),$$

This shows that q_0 is in S^0 uniformly in z. From the second formula we get similarly

$$\partial_z^j q_0 = -q_0 \sum_{k<j} \binom{j}{k} (\partial_z^{j-k} p_0)(\partial_z^k q_0)$$

$$= O(1) \sum_{k<j} \binom{j}{k} O(\langle\xi\rangle^{j-k}) O(\langle\xi\rangle^k) = O(\langle\xi\rangle^j);$$

this shows a first step in the estimates for the z-derivatives. The full system of estimates is obtained by application of a more general variant of (1.25) departing from $\partial_x^\beta \partial_\xi^\alpha \partial_z^j (p_0 q_0) = 0$.

Let χ and χ_1 denote C_0^∞ functions that are 1 on U and satisfy $\chi\chi_1 = \chi$. Since $q_0 \in S^{0,0}$,

$$r = 1 - p \circ (\chi_1 q_0) \in S^{-1,0} \text{ on the set where } \chi_1 = 1,$$

by (1.21) and Lemma 1.5. Now we can iterate and form a symbol

$$r_U \sim \sum_{k \geqq 1} (\chi r)^{ok};$$

here $(\chi r)^{\circ k}$ stands for $(\chi r) \circ (\chi r) \circ \cdots \circ (\chi r)$ with k factors; it belongs to $S^{-k,0}$ by the composition rule, so r_U exists in $S^{-1,0}$ by Lemma 1.9. Now $p \circ (\chi_1 q_0) \circ r_U \sim 1$ on U, so

$$(1.26) \qquad q_U \sim (\chi_1 q_0) \circ r_U \sim (\chi_1 q_0) \circ \textstyle\sum_{k \geqq 1} (\chi r)^{\circ k}$$

has the desired properties. □

q_U is a parametrix symbol for p on U. By suitable use of partitions of unity, this can be used to construct a parametrix of the operator with symbol p.

The theorem has consequences for symbols of other orders, e.g. for a symbol $p' = \mu^d [\xi, \mu]^m \circ p$, where p satisfies the hypotheses in the theorem.

2. Kernel and trace expansions on boundaryless manifolds

2.1. Kernel expansions for weakly polyhomogeneous ψdo's

The aim of the theory is to produce asymptotic expansions. We obtain these for *polyhomogeneous* symbols, cf. Definition 1.10. When $a(x, \xi, \mu)$ is homogeneous in (ξ, μ) for $|\xi| \geqq \varepsilon$, we denote by $a^h(x, \xi, \mu)$ the function that is homogeneous in (ξ, μ) for *all* $\xi \neq 0$ and coincides with $a(x, \xi, \mu)$ for $|\xi| \geqq \varepsilon$.

Theorem 2.1. *Let $p \sim \sum_{j \in \mathbb{N}} p_j$ in $S^{\infty,d}_{\mathrm{wphg}}(\mathbb{R}^n, \mathbb{R}^n, \Gamma)$ with degrees $\{m_j\}_{j \in \mathbb{N}}$ and μ-exponent d, and assume furthermore that p and the p_j with $m_j - d \geqq -n$ are in $S^{m',d'}$ for some $m' < -n$, some $d' \in \mathbb{R}$. Then* $\mathrm{OP}(p)$ *has a kernel $K_p(x, y, \mu)$ with an expansion on the diagonal*

$$(2.1) \qquad K_p(x,x,\mu) \sim \sum_{j=0}^{\infty} c_j(x) \mu^{m_j+n} + \sum_{k=0}^{\infty} [c'_k(x) \log \mu + c''_k(x)] \mu^{d-k} ,$$

for $|\mu| \to \infty$, uniformly for μ in closed subsectors of Γ. The coefficients $c_j(x)$ and $c'_{d-m_j-n}(x)$ are determined by $p_j(x, \xi, \mu)$ for $|\xi| \geqq 1$ (are "local"), while the $c''_k(x)$ are not in general determined by the homogeneous parts of the symbols (are "global").

For those values of j such that $p_j(x, \xi, \mu)$ is $O([\xi, \mu]^{m_j+n-\varepsilon}[\xi]^{-n+\varepsilon})$ with $\varepsilon > 0$, the contribution from p_j to the log *coefficient $c'_{d-m_j-n}(x)$ is 0, and the full contribution to the coefficient of μ^{m_j+n} is local.*

Proof. We can write $p(x, \xi, \mu) = \mu^d p'(x, \xi, \mu)$, where $p' = \mu^{-d} p$ satisfies the hypotheses with d replaced by 0 and m_j replaced by $m'_j = m_j - d$, cf. Definition 1.1. Then it suffices to show the theorem for p', for an expansion (2.1) of $K_{p'}$ with d resp. m_j replaced by 0 resp. m'_j gives an expansion (2.1) for K_p by multiplication by μ^d. Thus we can *assume that* $d = 0$ in the rest of the proof.

The hypotheses assure that all the symbols p_j and remainders $r_J = p - \sum_{0 \leqq j < J} p_j$ (including $r_0 = p$) are integrable in ξ for each μ; hence the operators they define have continuous kernels (cf. (1.17) ff.)

$$K_{p_j}(x, y, \mu) = \int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi} p_j(x, \xi, \mu)\, d\xi,$$
$$K_{r_J}(x, y, \mu) = \int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi} r_J(x, \xi, \mu)\, d\xi;$$

here $K_{r_0} = K_p$.

Consider first a remainder r_J; its kernel on the diagonal is given by

$$K_{r_J}(x,x,\mu) = \int_{\mathbb{R}^n} r_J(x,\xi,\mu) đ\xi. \tag{2.2}$$

By Theorem 1.12,

$$r_J(x,\xi,\mu) = \sum_{0 \leqq \nu < N} s_\nu(x,\xi)\mu^{-\nu} + O(\langle\xi\rangle^{m_J+N}\mu^{-N}), \text{ for any } N, \tag{2.3}$$

with $s_\nu \in S^{m_J+\nu}(\mathbb{R}^n, \mathbb{R}^n)$. For any given N, we can take J so large that

$$m_J + N < -n, \tag{2.4}$$

assuring integrability of each term for $|\xi| \to \infty$. Hence (2.2) and (2.3) give an expansion

$$K_{r_J}(x,x,\mu) = \sum_{0 \leqq \nu < N} c_{N,J,\nu}(x)\mu^{-\nu} + O(\mu^{-N}). \tag{2.5}$$

This will contribute to the c''_k. Note that the sum always begins with the power μ^0, regardless of how large N and J are taken.

Now we analyze the contribution from the homogeneous terms p_j.

Consider $p_j \in S^{m_j,0}$, homogeneous in (ξ,μ) of degree m_j for $|\xi| \geqq 1$. Split $\int_{\mathbb{R}^n} p_j\, đ\xi$ into three terms:

$$K_{p_j}(x,x,\mu) = \int_{\mathbb{R}^n} p_j\, đ\xi = \int_{|\xi| \geqq |\mu|} p_j\, đ\xi + \int_{|\xi| \leqq 1} p_j\, đ\xi + \int_{1 \leqq |\xi| \leqq |\mu|} p_j\, đ\xi. \tag{2.6}$$

First, $p_j(x,\xi,\mu)$ may be assumed homogeneous for $|\xi| \geqq 1$, so, for $|\mu| \geqq 1$,

$$\int_{|\xi| \geqq |\mu|} p_j(x,\xi,\mu) đ\xi = \mu^{m_j+n} \int_{|\xi| \geqq 1} (|\mu|/\mu)^{m_j+n} p_j(x,\xi,\mu/|\mu|) đ\xi \tag{2.7}$$

gives a term in μ^{m_j+n}. The coefficient of μ^{m_j+n} appears to depend on $\mu/|\mu|$, but if p_j is holomorphic it does *not*; see Lemma 2.3 below.

For the rest of $\int_{\mathbb{R}^n} p_j\, đ\xi$ we use the expansion from Theorem 1.12 again:

$$p_j(x,\xi,\mu) = \sum_{0 \leqq \nu < M} \mu^{-\nu} q_\nu(x,\xi) + R_M(x,\xi,\mu), \tag{2.8}$$

where $q_\nu(x,\xi) = \frac{1}{\nu!}\partial_z^\nu p(x,\xi,\frac{1}{z})|_{z=0}$ is in $S^{m_j+\nu}$ and homogeneous in ξ for $|\xi| \geqq 1$, and

$$R_M = O\big(\langle\xi\rangle^{m_j+M}\mu^{-M}\big)\,. \tag{2.9}$$

So the second integral

$$\int_{|\xi|\leqq 1} p_j(x,\xi,\mu)\,d\xi = \sum_{0\leqq\nu<M}\mu^{-\nu}\int_{|\xi|\leqq 1} q_\nu(x,\xi)\,đ\xi + O(\mu^{-M}) \tag{2.10}$$

gives further contributions to the $c_k''(x)\mu^{-k}$. Note that for each j, the series begins with μ^0 again. However, for each choice of N, and consequent choice of J such that (2.4) holds, we just have to apply (2.10) with $M \geqq N$ to the p_j with $j < J$. Together with (2.5) this gives $J+1$ contributions to each power $\mu^{-\nu}$, $\nu < N$, with a remainder $O(\mu^{-N})$.

In the third integral

$$\int_{1\leqq|\xi|\leqq|\mu|} p_j đ\xi, \tag{2.11}$$

p_j is homogeneous since $|\xi| \geqq 1$, and so are the q_ν in (2.8) and therefore R_M. So by (2.9),

$$R_M(x,\xi,\mu) = |\xi|^{m_j} R_M(x,\xi/|\xi|,\mu/|\xi|) \tag{2.12}$$

$$= O\big(|\xi|^{m_j+M}\mu^{-M}\big)\,, \quad |\xi| \geqq 1\,. \tag{2.13}$$

By extension by homogeneity, (2.13) holds for R_M^h for *all* $\xi \neq 0$.

Now expand (2.11) using (2.8) and (2.13), and taking $M > -m_j - n$. Because $q_\nu(x,\xi)$ is homogeneous in ξ of degree $m_j+\nu$ for $|\xi| \geqq 1$, one finds using polar coordinates:

$$\mu^{-\nu}\int_{1\leqq|\xi|\leqq|\mu|} q_\nu(x,\xi)đ\xi = \mu^{-\nu}c_\nu(x)\int_{1\leqq r\leqq|\mu|} r^{m_j+\nu+n-1}\,dr$$
$$= \begin{cases} \mu^{-\nu}c_\nu'(x)(|\mu|^{m_j+\nu+n}-1) & \text{if } m_j+\nu+n \neq 0, \\ \mu^{-\nu}c_\nu'(x)\log|\mu| & \text{if } m_j+\nu+n = 0; \end{cases}$$

and for R_M^h the homogeneity in (ξ,μ) and (2.13) imply

$$\int_{|\xi|\leqq|\mu|} R_M^h(x,\xi,\mu)\,đ\xi = c''(x)\mu^{m_j+n}, \qquad \int_{|\xi|\leqq 1} R_M^h(x,\xi,\mu)\,đ\xi = O(\mu^{-M}),$$

since $m_j + M > -n$. Thus, in view of Lemma 2.3 below,

$$\int_{1\leqq|\xi|\leqq|\mu|} p_j(x,\xi,\mu)\,đ\xi = \mu^{m_j+n}(c(x)+c'(x)\log\mu)$$
$$+\sum_{\nu=0}^{M-1} c_\nu(x)\mu^{-\nu} + O(\mu^{-M})\,, \tag{2.14}$$

with $c' = 0$ unless $m_j + n$ is an integer $\leqq 0$.

Together, (2.7), (2.10) and (2.14) give:

$$(2.15)\quad \int_{\mathbb{R}^n} p_j \, d\!\!{}^-\xi = c_j(x)\mu^{m_j+n} + c'_j(x)\mu^{m_j+n}\log\mu + \sum_{\nu=0}^{M-1} c_{j,\nu}(x)\mu^{-\nu} + O(\mu^{-M}),$$

where $c_j(x)$ and $c'_j(x)$ are determined from p_j for $|\xi| \geqq 1$ and the $c_{j,\nu}(x)$ need not be so.

The expansion (2.1) (with $d = 0$) is now obtained up to any error $O(\mu^{-N})$ by choosing J according to (2.4) and expanding the diagonal kernels of r_J and the p_j as in (2.5) and (2.15), with $M \geqq N$.

To show the last statement in the theorem, note that when p_j is $O([\xi,\mu]^{m_j+n-\varepsilon}[\xi]^{-n+\varepsilon})$,

$$p_j^h(x,\xi,\mu) = O(|\xi,\mu|^{m_j+n-\varepsilon}|\xi|^{-n+\varepsilon}),$$

by homogeneity. (For, $m_j + n - \varepsilon < 0$, and we can assume that $-n + \varepsilon < 0$.) Then

$$\begin{aligned}\int_{\mathbb{R}^n} p_j \, d\!\!{}^-\xi &= \int_{\mathbb{R}^n} p_j^h \, d\!\!{}^-\xi + \int_{|\xi|\leq 1} p_j \, d\!\!{}^-\xi - \int_{|\xi|\leq 1} p_j^h \, d\!\!{}^-\xi \\ &= \mu^{m_j+n} \int_{\mathbb{R}^n} (|\mu|/\mu)^{m_j+n} p_j^h(x,\xi,\mu/|\mu|) d\!\!{}^-\xi + O(\mu^{m_j+n-\varepsilon}) \qquad (2.16)\\ &= a_j(x)\mu^{m_j+n} + O(\mu^{m_j+n-\varepsilon}),\end{aligned}$$

in view of the homogeneity, the integrability at $\xi = 0$, and Lemma 2.3. Here $a_j(x)$ is determined from the homogeneous part of the symbol of p_j. A comparison of (2.15) and (2.16) shows that $c'_k(x) = 0$, $c_{j,\nu}(x) = 0$ for $\nu < m_j + n$, and $a_j(x) = c_j(x) + c_{j,m_j+n}(x)$. □

Remark 2.2. The $\log\mu$ terms, and the global terms $c''_k\mu^{d-k}$, come from the expansions of p and the p_j in powers of $z = \frac{1}{\mu}$. When any of these powers are absent, the corresponding coefficients c'_k and c''_k are absent.

The last statement in the theorem connects the present approach with that of [G1]; in fact the proof of the statement stems from there, where the estimate is satisfied by symbols "of regularity $\varepsilon - n$"; cf. [G1, Def. 3.1.1, Th. 3.3.4 and proof].

The following lemma was used in the proof:

Lemma 2.3. *If $f(z)$ is holomorphic in a sector and, as $r \to 0$,*

$$f(re^{i\theta}) = c(\theta)(re^{i\theta})^j \log^k(re^{i\theta}) + o(r^j \log^k(1/r)),$$

then $c(\theta)$ is independent of θ.

Proof. The function $z^{-j}\log^{-k}(z)f(z) = c(\theta) + o(1)$ is holomorphic in the given sector Γ. Let γ be any closed curve in Γ, and $t\gamma$ its contraction toward 0 by the factor t. Then

$$\begin{aligned}\int_{\gamma} c(\theta)dz &= \tfrac{1}{t}\int_{t\gamma} c(\theta)dz = \tfrac{1}{t}\int_{t\gamma} z^{-j}\log^{-k}(z)f(z)\,dz + \tfrac{1}{t}\int_{t\gamma} o(1)\,dz \\ &= 0 + o(1) \quad \text{as} \quad t \to 0\,.\end{aligned}$$

So $\int_\gamma c(\theta)\,d\theta = 0$ for all such γ, and $c(\theta)$ is a holomorphic function of θ alone, hence constant. □

2.2. *The ψdo resolvent on a compact manifold*

The definition of parameter-dependent ψdo's is extended to a compact manifold M in the usual way:

Definition 2.4. $P \in \mathrm{OP}(S^{m,d})$ *if and only if* $\varphi P \psi \in \mathrm{OP}(S^{m,d})$ *for all* φ, ψ *supported in a common coordinate neigborhood. For such P, in each local coordinate system, the full symbol* $\sigma(P) \in S^{m,d}(\mathbb{R}^n, \mathbb{R}^n, \Gamma)$ *is defined in local coordinates, modulo* $S^{-\infty,d}$ *(cf. (1.5)), and transforms with coordinate changes as in Proposition 1.22.* $\mathrm{OP}(S^{-\infty,d})$ *consists of operators with kernel* $K(x,y,\mu)$ *which is* C^∞ *in* (x,y), *holomorphic for* $\mu \in \Gamma$ $(|\mu| \geqq 1)$, *and has an expansion* $K \sim \sum K_j(x,y)\mu^{d-j}$ *with* $\partial_x^\alpha\partial_y^\beta(K - \sum_{0\leqq j<J} K_j\mu^{d-j}) = O(\mu^{d-J})$ *for* $|\mu| \to \infty$, *uniformly for* μ *in closed subsectors of* Γ.

The operator can map the sections of a C^∞ vector bundle E over M to another F.

Let $H^s(M,E)$ (or $H^s(E)$) denote the s-order L_2 Sobolev space of sections of E. It follows from standard results for ψdo's on compact manifolds that when $P \in \mathrm{OP}(S^{m,0})$, it maps $H^s(M,E)$ to $H^{s-m}(M,F)$ with norm uniformly bounded in μ for μ in closed subsectors of Γ, $|\mu| \geqq 1$ (cf. Definition 1.1). Since $S^{m,d} = \mu^d S^{m,0}$, one has when $P \in \mathrm{OP}(S^{m,d})$:

(2.17)
$$\|P\|_{\mathscr{L}(H^s(M,E),H^{s-m}(M,F))} \text{ is } O(\mu^d), \quad \text{for } \mu \text{ in closed subsectors of } \Gamma, |\mu| \geqq 1.$$

We now study parameter-elliptic ψdo's. Let m be a positive integer.

Definition 2.5. *A classical polyhomogeneous ψdo A of order m on M (acting in a bundle E of fiber dimension* $N > 0$ *over M), is said to be elliptic with parameter* $\mu \in \Gamma$ *if it is elliptic of order m, and the symbol* $\sigma(A) \sim a_m + a_{m-1} + \dots$ *in local coordinates is such that* $a_m(x,\xi) + \mu^m I$ *is invertible for* $\mu \in \Gamma$ *and* $|\xi| = 1$ *(i.e.,* $a_m(x,\xi)$ *has no eigenvalues in* $-\Gamma^m = \{-\mu^m \mid \mu \in \Gamma\}$*).*

Note that $\Gamma^m = \{\, \mu^m \mid \mu \in \Gamma \,\}$ cannot be all of $\mathbb{C} \setminus \{0\}$, but excludes at least one ray, so that holomorphic noninteger powers of $\lambda = -\mu^m$ can be defined there.

We can (and do) extend a_m smoothly to $|\xi| \leqq 1$ to have no eigenvalues in $-\Gamma^m \cup \{0\}$. By a standard procedure (see [S1], [Sh] or [G1]), we define symbols p_{-m-j} homogeneous of degree $-m-j$ in (ξ, μ) for $|\xi| \geqq 1$, so that

$$(\textstyle\sum a_{m-j} + \mu^m) \circ \sum p_{-m-k} \sim 1,$$

where $\circ$ is the symbol composition, cf. Definition 1.19. More precisely,

$$p_{-m} = (a_m + \mu^m)^{-1} \quad \text{for all} \quad \xi\,,$$

and for $j = 1, 2, \ldots,$ p_{-m-j} is a polynomial in p_{-m} and derivatives of $a_m, \ldots, a_{m-j}$, of degree $\geqq 2$ in p_{-m}. By Theorem 1.17 and Theorem 1.18 3°,

$$p_{-m} \in S^{-m,0}(\mathbb{R}^n, \mathbb{R}^n, \Gamma) \cap S^{0,-m}(\mathbb{R}^n, \mathbb{R}^n, \Gamma);$$

$$p_{-m-j} \in S^{-m-j,0}(\mathbb{R}^n, \mathbb{R}^n, \Gamma) \cap S^{m-j,-2m}(\mathbb{R}^n, \mathbb{R}^n, \Gamma), \quad j > 0. \tag{2.18}$$

Note that p_{-m-j} is a rational function of $\lambda = -\mu^m$ with coefficients homogeneous in ξ, so that the expansion of Theorem 1.12 gives only integer powers of $z^m = -\lambda^{-1}$: For $j = 1, 2, \ldots,$

$$\begin{aligned} p_{-m-j}(x, \xi, \mu) &= \sum_{2 \leqq \sigma < N} q_\sigma(x, \xi) \mu^{-m\sigma} + O(\langle \xi \rangle^{mN+m-j} \mu^{-Nm}) \\ &= \sum_{2 \leqq \sigma < N} (-1)^\sigma q_\sigma(x, \xi) \lambda^{-\sigma} + O(\langle \xi \rangle^{mN+m-j} \lambda^{-N})\,. \end{aligned} \tag{2.19}$$

(Here (2.19) can be verified e.g. by inserting the binomial expansions of the factors $(a - \lambda)^{-l} = (-\lambda)^{-l}(1 - a/\lambda)^{-l}$ in each term in p_{-m-j}.) The property that only the powers $\mu^{d-\nu}$ with $\nu = m\sigma$, σ integer, have nonzero Taylor coefficients in the expansions (1.7), is preserved under composition and change of variables (cf. Theorem 1.18, Proposition 1.22), so there are similar expansions for the various remainders that arise. For example, remainders in $S^{-\infty,-2m}$ have expansions in integer powers $\lambda^{-2}, \lambda^{-3}, \ldots$.

By Lemma 1.9, we construct a $p \in S^{-m,0} \cap S^{0,-m}$ with

$$p \sim \sum_{j=0}^{\infty} p_{-m-j} \quad \text{in} \quad S^{-m,0} \cap S^{0,-m}; \quad \text{here}$$

$$p - p_{-m} \sim \sum_{j=1}^{\infty} p_{-m-j} \quad \text{in} \quad S^{-m-1,0} \cap S^{m-1,-2m}\,. \tag{2.20}$$

Lemma 2.6. *With $-\lambda = \mu^m$, $(a - \lambda) \circ p - 1 \sim 0$ in $S^{-m,0} \cap S^{0,-m}$.*

Proof.

$$(a-\lambda)\circ p = [(a_m-\lambda)+\sum_1^\infty a_{m-j} \operatorname{mod} S^{-\infty,0}]$$
$$\circ[p_{-m}+\sum_1^\infty p_{-m-j} \text{ mod } S^{-\infty,-2m}].$$

Here $(a_m-\lambda)\circ S^{-\infty,-2m}\subset S^{-\infty,-m}$ and $S^{-\infty,0}\circ p_{-m}\in S^{-\infty,-m}$, and the other terms in the product are better. □

Now patch together such local coordinate operators in the usual way. Given a cover $\{U\}$, form

$$P=\sum \mathrm{OP}(\varphi_U p_U)\psi_U$$

where p_U is a symbol p as in (2.20) in the local coordinates in U, $0\leqq \varphi_U\in C_0^\infty(U)$, $\sum\varphi_U\equiv 1$, and $\psi_U\in C_0^\infty(U)$ with $\psi_U\equiv 1$ in a neighborhood of $\operatorname{supp}(\varphi_U)$. In local coordinates in U,

$$\sigma(\mathrm{OP}(\varphi_U p_U)\psi_U)\sim\varphi_U p_U$$

so by Proposition 1.22 on coordinate changes, we have in any local trivialization:

$$\sigma(P)\sim\sum_0^\infty p_{-m-j} \quad \text{in } S^{-m,0}\cap S^{0,-m},$$
$$\sigma(P)-p_{-m} \sim\sum_1^\infty p_{-m-j} \quad \text{in } S^{-m-1,0}\cap S^{m-1,-2m}.$$

By Lemma 2.6, $(A-\lambda)P=I-R$ with $R\in\mathrm{OP}(S^{-\infty,-m})$. In particular, the L_2 operator norm of R is $O(\lambda^{-1})$ (cf. (2.17)), so $I-R$ is invertible for large μ in Γ, and

$$(A-\lambda)^{-1}=P+P\sum_1^\infty R^j .$$

Since $R\in\mathrm{OP}(S^{-\infty,-m})$ and $P\in\mathrm{OP}(S^{0,-m})$, $PR^j\in\mathrm{OP}(S^{-\infty,-(1+j)m})$, and so $P+P\sum_{1\leqq j<J}R^j$ gives an expansion of $(A-\lambda)^{-1}$ in powers of λ^{-1}, up to an operator with a C^∞ kernel that is $O(\lambda^{-1-J})$. The parametrix P gives the (uniquely determined) "local" terms of the expansion of $(A-\lambda)^{-1}$, plus some "global" terms depending on arbitrary choices in the construction. The terms $PR, PR^2,\ldots$ give further global contributions.

Theorem 2.1 applies readily to the terms PR^j, but to P it applies directly only when $m>n$. When $m\leqq n$, we study instead a suitable power of the resolvent (or derivative), equally useful in functional calculus. Let k be a positive integer such that $km>n$. Then

$$(A-\lambda)^{-k}=\left(P+P\sum_{j\geqq 1}R^j\right)^k=P^k+\sum_{l\geqq 1}PR_{(l)},$$

where $PR_{(l)}$ is in $S^{-\infty,-km-lm}$, and

$$P^k = \mathrm{OP}(p^k_{-m} + p'), \quad \text{with}$$
$$p^k_{-m} \in S^{-km,0} \cap S^{0,-km}, \quad p' \sim \sum_{j \geqq 1} p'_j \text{ in } S^{-km-1,0} \cap S^{m-1,-(k+1)m};$$

the symbols $p^k_{-m} = (p_{-m})^k$ and p'_j being homogeneous in (ξ, μ) of degree $-km$ resp. $-km - j$ for $|\xi| \geqq 1$, and rational functions of λ.

By Theorem 2.1 and its proof one can now obtain a complete expansion of the diagonal kernel and the trace of $(A - \lambda)^{-k}$, in a series of terms $\lambda^{-j/m}$, λ^{-l} and $\lambda^{-l} \log \lambda$. The resulting expansion of $(A - \lambda)^{-k}$ has been shown by Agranovich in [Agr], where it is deduced from the pole analysis of the zeta function of A in [S1]. We shall here show a more general result, taking advantage of the fact that the present methods allow a pseudodifferential factor Q in front of $(A - \lambda)^{-k}$.

Theorem 2.7. *Let A and Q be classical polyhomogeneous ψdo's of order m (positive integer) resp. $\omega \in \mathbb{R}$ in a C^∞ vector bundle E over a compact n-dimensional manifold M. Assume that A is elliptic with parameter $\mu \in \Gamma$. Then for $\lambda \in -\Gamma^m$ and k with $-km + \omega < -n$, the kernel $K(x, y, \lambda)$ of $Q(A - \lambda)^{-k}$ satisfies on the diagonal:*

$$K(x,x,\lambda) \sim \sum_{j \in \mathbb{N}} c_j(x) \lambda^{\frac{n+\omega-j}{m} - k} + \sum_{l \in \mathbb{N}} (c'_l(x) \log \lambda + c''_l(x)) \lambda^{-l-k}, \tag{2.21}$$

for $|\lambda| \to \infty$, uniformly in closed subsectors of Γ. The coefficients $c_j(x)$ and $c'_l(x)$ are determined from the symbols $a \sim \sum_{j \in \mathbb{N}} a_{m-j}$ and $q \sim \sum_{j \in \mathbb{N}} q_{\omega-j}$ in local coordinates, while the $c''_l(x)$ are in general globally determined.

If Q is a differential operator (in particular if $Q = I$), $c'_0(x) = 0$ and the complete coefficient of λ^{-k} is locally determined.

As a consequence, one has for the trace:

$$\mathrm{Tr}Q(A - \lambda)^{-k} \sim \sum_{j \in \mathbb{N}} c_j \lambda^{\frac{n+\omega-j}{m} - k} + \sum_{l \in \mathbb{N}} (c'_l \log \lambda + c''_l) \lambda^{-k-l}\,; \tag{2.22}$$

where the coefficients are the integrals over M of the traces of the coefficients defined in (2.21).

Proof The symbol q of Q is in $S^{\omega,0}$ and weakly polyhomogeneous, cf. Examples 1.2 and 1.11. It follows from the preceding analysis plus the fact that the derivatives of p^k_{-m} contain $(a_m - \lambda)^{-1}$ in powers $\geqq 2$, that

$$Q(A - \lambda)^{-k} = \left(QP^k + \sum_{l \geqq 1} QPR_{(l)} \right); \text{ here}$$

$QPR_{(l)} \in \mathrm{OP}(S^{-\infty,-(l+1)m})$, and $QP^k = \mathrm{OP}(qp^k_{-m} + p'')$ in local trivializations,

with $\quad qp^k_{-m} \in S^{\omega-km,0} \cap S^{\omega,-km}, \quad p'' \sim \sum_{j \geqq 1} p''_j$ in $S^{\omega-km-1,0} \cap S^{\omega+m-1,-(k+1)m};$

where the symbols $q_{\omega-l}p^k_{-m}$ ($l \in \mathbb{N}$) and p''_j are homogeneous in (ξ, μ) of degree $\omega - l - km$ resp. $\omega - km - j$ for $|\xi| \geqq 1$, and depend on μ as rational functions of $\lambda = -\mu^m$.

The expansions of $q_{\omega-l}p^k_{-m}$ and p''_j according to Theorem 1.12 are then expansions in integer powers of λ (cf. (2.19) ff.); this also holds for the terms $QPR_{(l)}$ and the remainders $r_J = p'' - \sum_{1\leqq j<J} p''_j$.

Hence, applying Theorem 2.1 to $QPR_{(l)}$, we find for its kernel the asymptotic expansion for $|\lambda| \to \infty$:

$$K_{QPR_{(l)}}(x,x,\lambda) \sim \sum_{\sigma\in\mathbb{N}} c_{l,\sigma}(x)\lambda^{-(l+k)-\sigma}.$$

There is no difficulty in adding up the estimates for all l obtained in this way, since the starting power $-(l+k)$ goes to $-\infty$ for $l \to \infty$. Since $l = 1, 2, \ldots$, the highest power appearing here is λ^{-k-1}.

By Theorem 2.1 and Remark 2.2, the kernel corresponding to $p'' \in S^{\omega+m-1,-(k+1)m}$ has an expansion

$$K_{p''}(x,x,\lambda) = \sum_{j=1}^{\infty} c_j(x)\lambda^{-k+\frac{\omega-j+n}{m}} + \sum_{j=1}^{\infty}(c'_j(x)\log\lambda + c''_j(x))\lambda^{-k-j}.$$

And $qp^k_{-m} \in S^{\omega,-km}$ gives a kernel with a similar expansion, but starting with $j = 0$.

Collecting the various contributions, we find the first statement in the theorem.

Note that the only contributions to the coefficient of $\lambda^{-k}\log\lambda$, and to possibly global parts of the coefficient of λ^{-k}, come from qp^k_{-m}. The homogeneous extension of p^k_{-m} from $|\xi| \geqq 1$ to $|\xi| > 0$, $(p^k_{-m})^h = (a^h_m - \lambda)^{-k}$, is continuous at $\xi = 0$, so when Q is a differential operator (with polynomial symbol $q(x,\xi) = \sum_{0\leqq j\leqq\omega} q_{\omega-j}(x,\xi)$),

$$\int_{\mathbb{R}^n} qp^k_{-m}\,d\xi = \int_{\mathbb{R}^n} q(a^h_m - \lambda)^{-k}\,d\xi + \int_{|\xi|\leqq 1} q[(a_m - \lambda)^{-k} - (a^h_m - \lambda)^{-k}]\,d\xi,$$

where the first integral gives $\sum_{0\leqq j\leqq\omega} b_j(x)\lambda^{-k+\frac{\omega-j+n}{m}}$ with local coefficients b_j, and the second integral is $O(\lambda^{-k-1})$, since the integrand is so — as is seen e.g. from the calculation

$$\frac{1}{(a_m-\lambda)^k} - \frac{1}{(a^h_m-\lambda)^k} = \sum_{0\leqq j<k}\frac{1}{(a_m-\lambda)^{k-j-1}}\left[\frac{1}{a_m-\lambda} - \frac{1}{a^h_m-\lambda}\right]\frac{1}{(a^h_m-\lambda)^j},$$

where

$$\frac{1}{a_m-\lambda} - \frac{1}{a^h_m-\lambda} = \frac{1}{a_m-\lambda}[a^h_m - a_m]\frac{1}{a^h_m-\lambda} \text{ is } O(\lambda^{-2}).$$

This shows that when Q is a differential operator, $c'_0 = 0$ and the coefficient of λ^{-k} is local.

The expansion of the trace follows since the trace of the operator is the integral in x of the diagonal value of the (matrix trace of the) kernel. □

If Q and A are both differential operators, $a_{m-j} = a^h_{m-j}$ for all j, so all the terms $q_{\omega-l}p^k_{-m}$ and p''_j are strictly homogeneous and smooth; the splitting as in (2.6) is unnecessary, no logarithms occur, and all coefficients are locally determined. This is the most classical and well-known case (see e.g. Agmon and Kannai [AK]); here the QPR^j contribute with $O(\lambda^{-N})$ (any N), so the coefficients in the asymptotic expansion are fully determined from the symbols.

The expansion in Theorem 2.7 of the kernel of $Q(A-\lambda)^{-k}$ at $x = y$ determines the singularities of the meromorphic extension of the kernel of QA^{-s} (at $x = y$) and also, when the eigenvalues of $a_m(x,\xi)$ all lie in $Re\lambda > 0$, gives an expansion for the kernel of Qe^{-tA} (at $x = y$) as $t \to 0$. These are obtained by use of the relevant Cauchy integrals (see (3.36) below), as e.g. in [GS1]. The logarithmic terms in (2.21) and (2.22) correspond to double poles of the diagonal kernel resp. trace of $\Gamma(s)QA^{-s}$, and to logarithmic terms in the expansion for $t \to 0$ of the diagonal kernel resp. trace of Qe^{-tA}.

3. The resolvent of the APS problem

3.1. The APS problem

On a compact n-dimensional C^∞ manifold X with boundary $\partial X = X'$, consider a first-order differential operator

$$P : C^\infty(E_1) \to C^\infty(E_2)$$

between sections of vector bundles over X. E_1 and E_2 have Hermitian metrics, and X has a smooth volume element, defining a Hilbert space structure on the sections. The restrictions of the E_i to the boundary X' are denoted E'_i. A neighborhood of X' in X has the form $X = X' \times [0, c]$, and there the E_i are isomorphic to the pull-backs of the E'_i. Let x_n denote the coordinate in $[0, c]$. Then our P is represented in $X' \times [0, c]$ as

$$P = \sigma(\partial_n + A + x_n P_1 + P_0), \; \partial_n = \partial/\partial x_n, \tag{3.1}$$

where σ is a unitary morphism from E'_1 to E'_2, independent of x_n, and A is a fixed elliptic first order differential operator on $C^\infty(E'_1)$, selfadjoint with respect to the Hermitian metric in E'_1 and the volume element $v(x', 0)dx'$ on X' induced by the element $v(x', x_n)dx'dx_n$ on X. The P_j are smooth differential operators of order $\leqq j$ (they can be taken arbitrary near X', but for larger x_n, P_1 is subject to the requirement that P be elliptic). Similarly, P^* has the form

$$P^* = (-\partial_n + A + x_n P'_1 + P'_0)\sigma^*; \tag{3.2}$$

and

$$(Pu, w)_X - (u, P^*w)_X = -(\gamma_0 u, \sigma^* \gamma_0 w)_{X'}, \tag{3.3}$$

where $\gamma_0 u = u|_{X'}$. ([APS] and many subsequent works, including [GS1], consider the "cylindrical" or "product" case where the P_j and P'_j are 0.)

By $V_>, V_{\geqq}, V_<$ or $V_{\leqq}$ we denote the subspaces of $L^2(E'_1)$ spanned by the eigenvectors of A corresponding to eigenvalues which are $> 0, \geqq 0, < 0$, or $\leqq 0$. We denote by V_R the span of eigenvectors belonging to eigenvalues of modulus $\leqq R$, so V_0 is the nullspace of A. The corresponding projections are denoted $\Pi_>$, etc. Then Π_R is an integral operator with C^∞ kernel, and the other projections are classical ψdo's of order 0. For example,

$$\Pi_> = \tfrac{1}{2}(A - \varepsilon + |A - \varepsilon|)|A - \varepsilon|^{-1}, \; 0 < \varepsilon < \lambda_1(A),$$

where $\lambda_1(A)$ is the smallest positive eigenvalue of A. The principal symbols of these projections are denoted $\pi_R, \pi_>$, etc.

We also define

$$(3.4) \quad A_\mu = (A^2 + \mu^2)^{\frac{1}{2}}, \text{ for } \mu \in \Gamma_0 = \{\, \mu \in \mathbb{C} \setminus \{0\} \mid |\arg \mu| < \pi/2 \,\}.$$

To get a well-posed boundary value problem for P, we consider a ψdo B in E'_1 of order 0 satisfying:

Assumption 3.1. B is an orthogonal projection commuting with A, of the form $B = \Pi_> + B_0$, where B_0 acts in V_R and vanishes on $V_R^\perp$ for some $R \geqq 0$.

Since V_R is spanned by a finite system of C^∞ sections, B is a classical polyhomogeneous ψdo with principal symbol $\pi_>$.

The most customary choice of B is to take $\Pi_>$ or $\Pi_{\geqq}$ (cf. [APS] and subsequent works, e.g. [G2], [Gi],...). Another interesting choice is to take for B_0 a projection onto a subspace of V_0.

Since the Calderón projector for P has principal symbol $\pi_>$, B is well-posed for P in the sense of [S4, Definition VI.3], and the boundary value problem

$$(3.5) \qquad Pu = f \text{ in } X, \quad B\gamma_0 u = 0 \text{ on } X',$$

defines a Fredholm operator P_B going from $D(P_B) = \{u \in H^1(X, E_1) \mid B\gamma_0 u = 0\}$ to $L_2(X, E_2)$, with regularity of solutions. (Note that $\{P, B\gamma_0\}$ is *overdetermined elliptic*, also called *injectively elliptic*.) It follows moreover from [S4, Ch. VI] in view of (3.3) that the adjoint of P_B (considered as an unbounded operator from $L_2(E_1)$ to $L_2(E_2)$) equals $(P^*)_{B'}$, where

$$(3.6) \qquad B' = B^\perp \sigma^*, \quad B^\perp = I - B.$$

B' is well-posed for P^*, and $(P^*)_{B'}$ is a Fredholm operator with regularity of solutions. One can also replace B' by $B'' = \sigma B^\perp \sigma^*$, which is an orthogonal projection in $L_2(E'_2)$, but B' is convenient for (3.8) below. (The Fredholm property and the hypoellipticity of P_B and $(P^*)_{B'}$ also follow from the constructions carried out below.)

Remark 3.2. Of particular interest is the case where, in addition to the above hypotheses, $\sigma A \sigma^* = -A$ and $\sigma^* = -\sigma$. Then in the situation described in Douglas-Woiciechowsky [DW, App. 1], the nullspace V_0 admits an orthogonal decomposition $V_0 = V_{0,+} \oplus V_{0,-}$ into two subspaces such that $\sigma : V_{0,+} \xrightarrow{\sim} V_{0,-}$. Denoting the corresponding orthogonal projections by $\Pi_{0,\pm}$, we can take $B_0 = \Pi_{0,+}$. In this case, $\sigma \Pi_> = \Pi_< \sigma$ and $\sigma \Pi_{0,+} = \Pi_{0,-}\sigma$, so

$$\begin{aligned} B'' &= \sigma B^\perp \sigma^* = \sigma(I - \Pi_> - \Pi_{0,+})(-\sigma) = -\sigma(\Pi_< + \Pi_{0,-})\sigma \\ &= \Pi_> + \Pi_{0,+} = B, \end{aligned}$$

and thus if P is formally selfadjoint, P_B *is a selfadjoint realization.*

The two realizations P_B and $P_B{}^* = (P^*)_{B'}$ together form a skew-adjoint (unbounded) operator $\mathscr{P}_{\mathscr{B}}$ in $L_2(E_1 \oplus E_2)$:

$$(3.7) \quad \mathscr{P}_{\mathscr{B}} = \begin{pmatrix} 0 & -P_B{}^* \\ P_B & 0 \end{pmatrix}, \text{ the realization of } \mathscr{P} = \begin{pmatrix} 0 & -P^* \\ P & 0 \end{pmatrix}$$

under the boundary condition on $u = \{u_1, u_2\}$,

$$(3.8) \qquad \mathscr{B}\gamma_0 u = 0, \text{ where } \mathscr{B} = (B \ B') : \begin{matrix} L_2(E_1') \\ \times \\ L_2(E_2') \end{matrix} \to L_2(E_1').$$

The system $\{\mathscr{P}, \mathscr{B}\gamma_0\}$ associated with $\mathscr{P}_{\mathscr{B}}$ is in fact *elliptic* in the traditional sense, however with $\mathscr{B}$ pseudodifferential. (One easily checks that the model problem for the principal symbol has existence of solutions in $\mathscr{S}(\overline{\mathbb{R}}_+)^{2N}$, hence also uniqueness for dimensional reasons.)

Our construction of functions of P_B will be based on a study of the resolvent.

Since $\mathscr{P}_{\mathscr{B}}$ is skew-adjoint, the inverse $(\mathscr{P}_{\mathscr{B}} + \mu)^{-1}$ exists (as a bounded operator in $L_2(E_1 \oplus E_2)$) for all $\mu \in \pm\Gamma_0$ (cf. (3.4)); it can be written in detail as

$$\mathscr{R} = (\mathscr{P}_{\mathscr{B}} + \mu)^{-1} = \begin{pmatrix} \mu R_1 & P_B{}^* R_2 \\ -P_B R_1 & \mu R_2 \end{pmatrix}, \quad \text{where}$$

$$(3.9) \qquad R_1 = (P_B{}^* P_B + \mu^2)^{-1}, \quad R_2 = (P_B P_B{}^* + \mu^2)^{-1}.$$

(Since most of the operators considered in the following depend on μ, we shall usually not indicate this explicitly.) In the following, we construct $\mathscr{R}$ from scratch for large μ.

We want to study the asymptotic behavior of

$$\operatorname{Tr}[\varphi \partial_\mu^m (\mathscr{P}_{\mathscr{B}} + \mu)^{-1}], \quad \text{for } |\mu| \to \infty, \mu \in \pm\Gamma_0,$$

where φ is a morphism in $E_1 \oplus E_2$, and m is taken $\geqq n$ so that the operator is of trace class (shown below). (Note that $\partial_\mu^m(\mathscr{P}_{\mathscr{B}}+\mu)^{-1} = (-1)^m m!(\mathscr{P}_{\mathscr{B}}+\mu)^{-m-1}$.) We shall denote

$$E_1 \oplus E_2 = E, \quad E_1' \oplus E_2' = E'.$$

In order to analyze the inverse (3.9), we shall compare it with the inverse in the product case. Define

$$\text{(3.10)} \quad P^0 = \sigma(\partial_n + A), \quad P^{0\prime} = (-\partial_n + A)\sigma^*, \quad \text{so } P^{0\prime}P^0 = D_n^2 + A^2.$$

They have a meaning on $X^0 = X' \times \mathbb{R}_+$; and $P^{0\prime}$ is the adjoint of the unbounded operator $P^0 : L_2(E_1^0) \to L_2(E_2^0)$, where the E_i^0 are the liftings of the E_i' to X^0, and the product measure is used. We denote $\begin{pmatrix} 0 & -P^{0\prime} \\ P^0 & 0 \end{pmatrix} = \mathscr{P}^0$, and observe that

$$\text{(3.11)} \quad (\mathscr{P}^0+\mu)^{-1} = \begin{pmatrix} \mu(D_n^2+A^2+\mu^2)^{-1} & (-\partial_n+A)(D_n^2+A^2+\mu^2)^{-1}\sigma^* \\ -\sigma(\partial_n+A)(D_n^2+A^2+\mu^2)^{-1} & \mu\sigma(D_n^2+A^2+\mu^2)^{-1}\sigma^* \end{pmatrix}.$$

An important ingredient in the resolvent $(\mathscr{P}_{\mathscr{B}}+\mu)^{-1}$ is the following *Poisson operator*

$$\text{(3.12)} \qquad K_{\mathscr{B}}^0 = \begin{pmatrix} 1 & 0 \\ 0 & \sigma \end{pmatrix} e^{-x_n A_\mu} \begin{pmatrix} B + \mu^{-1}(A_\mu + A)B^\perp \\ \mu^{-1}(A_\mu - A)B + B^\perp \end{pmatrix}.$$

Since A commutes with B, a direct calculation shows that

$$\text{(3.13)} \qquad \mathscr{B}\gamma_0 K_{\mathscr{B}}^0 = I \quad \text{and} \quad (\mathscr{P}^0+\mu)K_{\mathscr{B}}^0 = 0.$$

In other words, $K_{\mathscr{B}}^0 : v \mapsto u$ solves the semi-homogeneous problem for $\mathscr{P}^0+\mu$ with $(\mathscr{P}^0+\mu)u = 0$ on the cylinder X^0 and $\mathscr{B}\gamma_0 u = v$.

We shall use $K_{\mathscr{B}}^0$ together with an interior parametrix Q_+, constructed as follows. Extend $\mathscr{P}$ to a bundle $\widetilde{E} = \widetilde{E}_1 \oplus \widetilde{E}_2$ over $\widetilde{X} = X \cup (X' \times]-1, 0[\,)$, and define a resolvent-parametrix Q' in the usual way such that for u in $C_0^\infty(\widetilde{E})$,

$$(\mathscr{P}+\mu)Q'u = u - \mathscr{S}u\,,$$

with $\mathscr{S}$ of order $-\infty$. Since $\sigma(\mathscr{P})+\mu$ is the symbol of a first order elliptic differential operator in one more variable, Q' can be constructed so as to be a strongly polyhomogeneous ψdo of degree -1 and $\mathscr{S}$ is strongly polyhomogeneous of degree $-\infty$ (cf. Theorem 1.16; one can also use Theorem 1.23). In particular, $\mathscr{S}$ is an integral operator with a smooth kernel that together with its derivatives is $O(\mu^{-N})$ for all N, on compact sets in $\widetilde{X}$. We can take these operators to be holomorphic in $\pm\Gamma_0$ except for a small neighborhood of 0; then the estimates hold for μ in closed subsectors of $\pm\Gamma_0$. For operators S on $\widetilde{X}$ we generally write

$$\text{(3.14)} \qquad S_+ u = r^+ S e^+ u$$

where e^+u is the extension of u with $e^+u(x',x_n) = 0$ for $x_n < 0$, and r^+ denotes restriction to X. Since $\mathscr{P}$ is a differential operator,

$$(\mathscr{P} + \mu)Q'_+u = u - \mathscr{S}_+u\,, \text{ for } u \in C^\infty(E),$$

where $\mathscr{S}_+$ is an integral operator on X with a smooth kernel that together with its derivatives is $O(\mu^{-N})$ for all N. Thus for sufficiently large μ in each closed subsector of $\pm\Gamma_0$, $I - \mathscr{S}_+$ is invertible with an inverse $(I - \mathscr{S}_+)^{-1} = I + \sum_{j\geqq 1}(\mathscr{S}_+)^j$, where $\mathscr{S}'_+ = \sum_{j\geqq 1}(\mathscr{S}_+)^j$ is of the same type as $\mathscr{S}_+$. We take $Q_+ = Q'_+ + Q'_+\mathscr{S}'_+$ (where $Q'_+\mathscr{S}'_+$ is of the same type as $\mathscr{S}_+$); it satisfies

$$(\mathscr{P} + \mu)Q_+ = I, \text{ for large enough } \mu. \tag{3.15}$$

Now a rough parametrix for $\mathscr{P}_\mathscr{B} + \mu$ is

$$\mathscr{R}' = Q_+ - \chi K^0_\mathscr{B}\mathscr{B}\gamma_0 Q_+ =: Q_+ - G_1, \tag{3.16}$$

where χ is a cut-off function (with $\chi(x_n) = 1$ for $x_n < \frac{1}{4}c$ and $\chi(x_n) = 0$ for $x_n > \frac{1}{2}c$, say). Indeed, by (3.13), $\mathscr{R}'$ maps into the domain of $\mathscr{P}_\mathscr{B}$, and by (3.15), we have for large enough μ,

$$\begin{aligned}(\mathscr{P} + \mu)\mathscr{R}' &= (\mathscr{P} + \mu)Q_+ - (\mathscr{P} + \mu)\chi K^0_\mathscr{B}\mathscr{B}\gamma_0 Q_+ \\ &= I - ([\mathscr{P},\chi] + \chi(\mathscr{P} - \mathscr{P}^0))K^0_\mathscr{B}\mathscr{B}\gamma_0 Q_+ =: I - G_2; \\ \text{here} \quad G_2 &= (x_n\mathscr{P}_1 + \mathscr{P}_0)K^0_\mathscr{B}\mathscr{B}\gamma_0 Q_+,\end{aligned} \tag{3.17}$$

with differential operators $\mathscr{P}_j$ of order j with smooth coefficients vanishing for $x_n > \frac{1}{2}c$.

The exact inverse $\mathscr{R}$ of $\mathscr{P}_\mathscr{B} + \mu$ can then be described by

$$\mathscr{R} = \mathscr{R}'(I - G_2)^{-1} = (Q_+ - G_1)(I - G_2)^{-1}, \tag{3.18}$$

whenever $I - G_2$ is invertible. We shall show that this holds for large μ, and leads to a constructive expression for $\mathscr{R}$. For this purpose, we analyze the various factors in (3.16) and (3.17). Let us denote

$$K_0 = e^{-x_nA_\mu}, \quad S_\mathscr{B} = \begin{pmatrix} B + \mu^{-1}(A_\mu + A)B^\perp \\ \mu^{-1}(A_\mu - A)B + B^\perp \end{pmatrix}\mathscr{B}, \quad T_0 = \gamma_0 Q_+,$$

$$K_1 = \chi\begin{pmatrix} 1 & 0 \\ 0 & \sigma \end{pmatrix}K_0, \quad K_2 = (x_n\mathscr{P}_1 + \mathscr{P}_0)\begin{pmatrix} 1 & 0 \\ 0 & \sigma \end{pmatrix}K_0. \tag{3.19}$$

Here K_0 goes from $C^\infty(E'_1)$ to $C^\infty(E^0_1)$, $S_\mathscr{B}$ goes from $C^\infty(E')$ to $C^\infty(E'_1 \oplus E'_1)$, T_0 goes from $C^\infty(E)$ to $C^\infty(E')$, and K_1 and K_2 go from $C^\infty(E'_1 \oplus E'_1)$ to $C^\infty(E)$. Then

$$G_1 = K_1S_\mathscr{B}T_0, \quad G_2 = K_2S_\mathscr{B}T_0. \tag{3.20}$$

In the terminology of [BM] and [G1], the K_j are parameter-dependent Poisson operators and T_0 is a parameter-dependent trace operator of class 0 (trace operators of class 0 are well-defined on L_2), but their usage entered elliptic theory much earlier, cf. [S2], [H1]. We shall need some general information on these operator types.

3.2. *Polyhomogeneous boundary operators*

The operators K_j and T_0 have the following structure, in local coordinates at $\partial X = X'$ (with interpretations as oscillatory integrals), defining *Poisson* resp. *trace* operators:

$$(3.21)\qquad (Kv)(x) = \mathrm{OPK}(\tilde{k})v = \int_{\mathbb{R}^{n-1}} e^{ix'\cdot\xi'}\tilde{k}(x',x_n,\xi',\mu)\hat{v}(\xi')\,đ\xi',$$

$$(3.22)\qquad (Tu)(x') = \mathrm{OPT}(\tilde{t})u = \int_{\mathbb{R}^{n-1}} e^{ix'\cdot\xi'}\int_0^\infty \tilde{t}(x',y_n,\xi',\mu)\acute{u}(\xi',y_n)\,dy_n\,đ\xi',$$

where $\acute{u}(\xi',y_n) = \mathscr{F}_{y'\to\xi'}u(y)$ ($\mathscr{F}$ is the Fourier transform). Note that with respect to the tangential variables these are ψdo's, whereas for the Poisson operator K there is a multiplication with a function of x_n and for the trace operator T there is a scalar product with a function of x_n (the dual situation). The functions $\tilde{k}(x,\xi',\mu)$ and $\tilde{t}(x,\xi',\mu)$ are called the *symbol-kernels*, and the functions $k(x',\xi,\mu)$ and $t(x',\xi,\mu)$ obtained from them by applying $\mathscr{F}_{x_n\to\xi_n}e^+$ are called the *symbols*. (Recall that e^+ denotes extension by 0 for $x_n \leqq 0$.)

Our symbols k and t are polyhomogeneous in (ξ',ξ_n,μ) (for $|\xi'| \geqq 1$), whereas the symbol-kernels $\tilde{k}$ and $\tilde{t}$ have a poly-quasihomogeneity in (x_n,ξ',μ).

The Poisson and trace symbols and symbol-kernels to be considered here are of the *strongly* polyhomogeneous type where, as in Definition 1.15ff., the parameter-dependent symbols are like standard symbols in one more (tangential) variable $z_{n+1} \in \mathbb{R}$, in such a way that they are constant in z_{n+1} but the dual variable ζ_{n+1} plays the role of μ. This means that μ enters on equal footing with the variables in ξ', and e.g. homogeneities hold for $|\xi',\mu| \geqq 1$, not just $|\xi'| \geqq 1$, and estimates are valid with respect to $\langle\xi',\mu\rangle$, not just $\langle\xi'\rangle$ alone.

In comparison, the book [G1] treats irregular boundary symbol classes with a complicated distinction betweeen the (ξ',μ)-behavior and the ξ'-behavior. But for the present problems, we can formulate our calculations in such a way that this distinction is not needed for those operators that reach into the interior of X, only for the ψdo's on X'.

Definition 3.3. *A strongly poly-quasihomogeneous symbol-kernel with parameter, $\tilde{p}(x',x_n,\xi',\mu)$, is a C^∞ function of $(x,\xi',\mu) \in \overline{\mathbb{R}}^n_+ \times \mathbb{R}^{n-1} \times \Gamma$ having an asymptotic expansion in functions quasihomogeneous for $|\xi',\mu| \geqq 1$. Setting $\zeta = (\xi',\mu)$, this means*

$$\tilde{p}(x,\zeta) \sim \sum_{j\in\mathbb{N}} \tilde{p}_{m-j}(x,\zeta),$$

where

$$\tilde{p}_{m-j}(x',\tfrac{x_n}{a},a\zeta) = a^{m+1-j}\,\tilde{p}_{m-j}(x',x_n,\zeta)\quad \text{for } a>0,\ |\zeta| \geqq 1,$$

with

$$\sup_{x_n} |\partial_{x'}^{\beta} \partial_{\zeta}^{\alpha} x_n^k \partial_{x_n}^l \tilde{p}_{m-j}(x, \zeta)| = O(\langle \zeta \rangle^{m+1-|\alpha|-k+l}),$$

$$\sup_{x_n} \left| \partial_{x'}^{\beta} \partial_{\zeta}^{\alpha} x_n^k \partial_{x_n}^l \left[\tilde{p}(x, \zeta) - \sum_{j<J} \tilde{p}_{m-j}(x, \zeta) \right] \right| = O(\langle \zeta \rangle^{m+1-J-|\alpha|-k+l}), \tag{3.23}$$

for all indices $\alpha \in \mathbb{N}^n$, $\beta \in \mathbb{N}^{n-1}$, $k, l, J \in \mathbb{N}$. *The term* $\tilde{p}_m$ *is called the principal symbol-kernel, and* $\tilde{p}$ *is called of degree* m. *The estimates are to hold uniformly for* x' *in compact subsets of* $\mathbb{R}^{n-1}$ *and* μ *in closed subsectors of* Γ, *and* $\tilde{p}$ *and* $\tilde{p}_j$ *are assumed holomorphic in* $\mu \in \mathring{\Gamma}$, *for* $|\xi', \mu| \geqq 1$.

The *degree* convention may seem strange at first sight, but it is consistent with the degree of the associated principal *symbol* $p_m(x', \zeta, \xi_n) = F_{x_n \to \xi_n} e^+ \tilde{p}_m$, which is truly homogeneous in (ζ, ξ_n) of degree m for $|\zeta| \geqq 1$ (this avoids the general ambiguity of how to associate a degree with a quasihomogeneous function). We shall call the symbol-kernels satisfying Definition 3.3, and the associated symbols, and the operators defined from them by (3.21), (3.22), *strongly polyhomogeneous* (sphg).

When the estimates hold globally in $x' \in \mathbb{R}^{n-1}$, the symbol-kernels, symbols and operators will be said to be *globally estimated*; these are treated systematically in [GK]. See e.g. [G1] for the definition of operators on manifolds.

We collect some facts on these operators and their composition rules in the Appendix. For the operators in (3.19), we find:

Lemma 3.4. *Let* m, m_1 *and* m_2 *be integers* $\geqq 0$, *and let* $\mu \in \pm\Gamma_0$.

The operator $A_\mu = (A^2 + \mu^2)^{\frac{1}{2}}$ *and its inverse are sphg* ψ*do's on* X' *of degree* 1 *resp.* -1.

The operator $\partial_\mu^m T_0$ *is an sphg trace operator of class 0, of degree* $-1-m$.

Each of the operators $\partial_\mu^m K_j$ *is an sphg Poisson operator of degree* $-1-m$.

Furthermore, $\partial_\mu^{m_1} T_0 \partial_\mu^{m_2} K_j$ *is an sphg* ψ*do on* X', *of degree* $-1-m_1-m_2$.

Proof. By the construction in [S1], the symbols of A_μ and A_μ^{-1} satisfy Definition 1.15 with $m = 1$ resp. -1.

The statement on $\partial_\mu^m T_0$ follows from Lemma A.1, (ii) and (vii).

For K_0 we observe that with $\mathcal{Q} = (D_n^2 + A^2 + \mu^2)^{-1}$ on the cylinder $\widetilde{X}^0 = X' \times \mathbb{R}$,

$$\begin{aligned} r^+ \mathcal{Q} \tilde{\gamma}_0^* v &= r^+ (D_n^2 + A^2 + \mu^2)^{-1} \tilde{\gamma}_0^* v = r^+ F_{\xi_n \to x_n}^{-1} \frac{1}{\xi_n^2 + A^2 + \mu^2} (v \otimes 1) \\ &= r^+ F_{\xi_n \to x_n}^{-1} \left(\frac{1}{A_\mu + i\xi_n} + \frac{1}{A_\mu - i\xi_n} \right) \left(\frac{1}{2A_\mu} v \otimes 1 \right) \\ &= r^+ e^{-x_n A_\mu} \frac{1}{2A_\mu} v = K_0 \frac{1}{2A_\mu} v, \end{aligned} \tag{3.24}$$

for $v \in C^\infty(E_1')$, cf. Lemma A.1 (iii). The calculations are similar to those in Example A.2, except that we here use simple operator calculus, noting that A^2 is selfadjoint $\geqq 0$ in $L_2(E_1')$ so that A_μ has spectrum in a subsector of $\{\lambda \in \mathbb{C} \mid |\arg \lambda| < \frac{\pi}{2}\}$. Then by Lemma A.1 (iii), $K_0 \frac{1}{2A_\mu}$ is an sphg Poisson operator of degree -2. Since A_μ is sphg of degree 1, it follows by Lemma A.1 (v) that K_0 is an sphg Poisson operator of degree -1. The statements on derivatives follow from Lemma A.1 (vii). Now

$$K_1 = r^+\chi \begin{pmatrix} 1 & 0 \\ 0 & \sigma \end{pmatrix} \mathcal{Q}\tilde{\gamma}_0^* 2A_\mu, \quad K_2 = r^+(x_n\mathcal{P}_1 + \mathcal{P}_0) \begin{pmatrix} 1 & 0 \\ 0 & \sigma \end{pmatrix} \mathcal{Q}\tilde{\gamma}_0^* 2A_\mu;$$

these are treated similarly by Lemma A.1 (iii), (v) and (vii).

The last rule follows by Lemma A.1 (iv). □

Since $\mathcal{Q}$ is x_n-independent, the symbol-kernel of K_0 can easily be found by use of Lemma A.1. The principal symbol of $\mathcal{Q}$ is $(\xi_n^2 + a_1(x', \xi')^2 + \mu^2)^{-1}$, and then the principal symbol-kernel of K_0 is $\exp(-x_n(a_1^2 + \mu^2)^{\frac{1}{2}})$. Since $\mathcal{P}$ coincides principally with $\mathcal{P}^0$ on $\partial X = X'$, the principal symbol-kernel of T_0 can be found similarly from (3.11).

The mapping properties are described in Lemma A.3 and A.4.

Now consider $S_{\mathcal{B}}$. (OP$'$ indicates the ψdo definition applied with respect to $x' \in \mathbb{R}^{n-1}$, or $x' \in X'$.) Recall that $\mu \in \pm\Gamma_0$.

Proposition 3.5. *The operator* $S_{\mathcal{B}}$ *in* (3.19) *is in the class* OP$'(S^{0,0})$ *of Section 1,* $\partial_\mu^m S_{\mathcal{B}}$ *is in* OP$'(S^{-m,0} \cap S^{0,-m})$ *for* m *integer* $\geqq 0$, *and they are weakly polyhomogeneous.*

Proof. Write $S_{\mathcal{B}} = S'_{\mathcal{B}}\mathcal{B}$, where

$$S'_{\mathcal{B}} = \begin{pmatrix} B + \mu^{-1}(A_\mu + A)B^\perp \\ \mu^{-1}(A_\mu - A)B + B^\perp \end{pmatrix}.$$

Then it suffices to show that $S'_{\mathcal{B}}$ has the properties stated in the proposition. For $\mathcal{B}$ is μ-independent and classically polyhomogeneous of order 0, hence in $S^{0,0}$ and weakly polyhomogeneous by Examples 1.2 and 1.11, and the composition rule in Theorem 1.18 can then be applied. Also B and $B^\perp$ have the properties stated in the proposition. Now

$$\mu^{-1}(A_\mu + A)B^\perp = \mu^{-1}(A_\mu + A)(\Pi_{\leqq} - B_0),$$

$$\mu^{-1}(A_\mu - A)B = \mu^{-1}(A_\mu - A)(\Pi_{>} + B_0),$$

where B_0 acts in V_R, vanishes on $V_R^\perp$ and commutes with A; it can be expressed as a matrix acting on the eigenvectors in V_R. Let a be an eigenvalue for A with $|a| \leqq R$, then $\partial_z^k(a^2z^2 + 1)^{\frac{1}{2}}$ is uniformly bounded when

$|z| \leqq 1$ (z in a closed subsector), and the eigenvectors of A are C^∞. Then $\partial_z^k[(A^2z^2+1)^{\frac{1}{2}} \pm zA]B_0$ is uniformly in $S^{-\infty}$, so

$$\mu^{-1}(A_\mu \pm A)B_0 \in \mathrm{OP}'(S^{-\infty,0}).$$

For the remaining parts we observe:

$$\mu^{-1}(A_\mu + A)\Pi_\leqq = \mu^{-1}(A_\mu + A)(A_\mu - A)(A_\mu + |A|)^{-1}\Pi_\leqq = \mu(A_\mu + |A|)^{-1}\Pi_\leqq,$$

$$\mu^{-1}(A_\mu - A)\Pi_> = \mu^{-1}(A_\mu - A)(A_\mu + A)(A_\mu + |A|)^{-1}\Pi_> = \mu(A_\mu + |A|)^{-1}\Pi_>.$$

Since $\Pi_>$ and $\Pi_\leqq$ are in $S^0 \subset S^{0,0}$ and wphg (cf. Examples 1.2 and 1.11), we need only consider $\mu(A_\mu + |A|)^{-1} = (\mu A_\mu^{-1})A_\mu(A_\mu + |A|)^{-1}$. Here A_μ and A_μ^{-1} are sphg by Lemma 3.4, and $|A|$ is wphg.

Now $A_\mu(A_\mu + |A|)^{-1} = C^{-1}$, where $C = I + \frac{|A|}{A_\mu}$ has symbol in $S^{0,0}$ by Example 1.2, Theorem 1.16 and the product rule. Its principal symbol $1 + [|a_1|](a_1^2 + \mu^2)^{-\frac{1}{2}}$ has a bounded inverse (for μ in closed subsectors of $\pm\Gamma_0$, $|\mu| \geqq 1$), so Theorem 1.23 applies to show that C has a wphg parametrix C' with symbol in $S^{0,0}$; here $CC' = 1 - \mathscr{S}_1$ with $\mathscr{S}_1$ in $S^{-\infty,0}$. On the other hand, the inverse C^{-1} may be written

$$C^{-1} = \frac{A_\mu}{A_\mu + |A|} = \frac{A_\mu(A_\mu - |A|)}{A_\mu^2 - |A|^2} = \frac{1}{\mu^2}(A^2 + \mu^2 - A_\mu|A|),$$

so it is a ψdo with symbol in $S^{2,-2} + S^{0,0} + S^{1,-1}$, by the various rules. Then

$$C^{-1} - C' = C^{-1}(CC' + \mathscr{S}_1) - C^{-1}CC' = C^{-1}\mathscr{S}_1$$

has symbol in $(S^{2,-2} + S^{0,0} + S^{1,-1}) \cdot S^{-\infty,0} \subset S^{0,0}$, so C^{-1} has symbol in $S^{0,0}$.

Since $A_\mu^{-1} \in \mathrm{OP}'(S^{-1,0} \cap S^{0,-1})$, then $(A_\mu + |A|)^{-1} = A_\mu^{-1}C^{-1} \in \mathrm{OP}'(S^{-1,0} \cap S^{0,-1})$; it is wphg. Now

$$\partial_\mu^m\left(\frac{\mu}{A_\mu + |A|}\right) = \sum_{j+k-l=m} c_{jkl}(A_\mu + |A|)^{-j}A_\mu^{-k}\mu^l,$$

with $j,k,l,m \geqq 0$. Writing $l = l_1 + l_2$ with $0 \leqq l_1 \leqq j$ and $0 \leqq l_2 \leqq k$, we have

$$(A_\mu + |A|)^{-j}A_\mu^{-k}\mu^{l_1+l_2} = \left(\frac{\mu}{A_\mu + |A|}\right)^{l_1}\left(\frac{\mu}{A_\mu}\right)^{l_2}(A_\mu + |A|)^{l_1-j}A_\mu^{l_2-k}$$

$$\in OP'(S^{0,0} \cdot (S^{l_1-j,0} \cap S^{0,l_1-j}) \cdot (S^{l_2-k,0} \cap S^{0,l_2-k})) \subset OP'(S^{-m,0} \cap S^{0,-m}).$$

Thus $\partial_\mu^m(\mu(A_\mu + |A|)^{-1}) \in OP'(S^{-m,0} \cap S^{0,-m})$ and is wphg; this concludes the proof. □

The properties of the $\partial_\mu^m S_\mathscr{B}$ are needed because we take a derivative of the resolvent to get an operator of trace class; they are not implied by Lemma 1.5.

The mapping properties of the various operators are as follows.

Proposition 3.6. *The operators Q_+, $\mathscr{R}'$, K_j, S_B and T_0 (cf. (3.16)–(3.20)) and their μ-derivatives satisfy, for $m \in \mathbb{N}$, $s \geqq 0$,*

$$
\begin{aligned}
\partial_\mu^m Q_+ \text{ and } \partial_\mu^m \mathscr{R}' &: L_2(E) \to H^{m+1,\mu}(E),\\
\partial_\mu^m T_0 &: H^{s,\mu}(E) \to H^{s+m+\frac{1}{2},\mu}(E'),\\
\partial_\mu^m S_{\mathscr{B}} &: H^{s+\frac{1}{2},\mu}(E') \to H^{s+m+\frac{1}{2},\mu}(E_1' \oplus E_1'),\\
\partial_\mu^m K_1 \text{ and } \partial_\mu^m K_2 &: H^{s+\frac{1}{2},\mu}(E_1' \oplus E_1') \to H^{s+1+m,\mu}(E);
\end{aligned}
\tag{3.25}
$$

uniformly for μ in closed subsectors of $\pm\Gamma_0$, $|\mu| \geqq r$ (depending on the subsector).

Proof. It is standard that $\varrho_1 \partial_\mu^m Q_k \varrho_2 : L_2(\widetilde{E}) \to H^{m+k,\mu}(\widetilde{E})$, uniformly in μ, when Q_k is an sphg ψdo in $\widetilde{E}$ of degree $-k$, and ϱ_1 and $\varrho_2 \in C_0^\infty(\widetilde{X})$; this implies the statement for Q_+ constructed before (3.15).

By Lemma 3.4 and Proposition 3.5, $\partial_\mu^m K_j$ is an sphg Poisson operator of degree $-1-m$, $\partial_\mu^m S_{\mathscr{B}}$ is in $OP'(S^{-m,0} \cap S^{0,-m})$, and $\partial_\mu^m T_0$ is an sphg trace operator of degree $-1-m$. Then the statements in (3.25) follow from Lemmas A.3 and A.4 (applied in local trivializations). The property for $\mathscr{R}' = Q_+ - G_1$ follows since $G_1 = K_1 S_{\mathscr{B}} T_0$. □

3.3. The resolvent of the APS problem

Recall (3.18). In the construction of $\mathscr{R}$, we shall use the elementary

Lemma 3.7. *Let $\Phi : V \to W$ and $\Psi : W \to V$ be linear mappings. If $I - \Phi\Psi : W \to W$ is bijective, then $I - \Psi\Phi : V \to V$ is bijective, with*

$$(I - \Psi\Phi)^{-1} = I + \Psi(I - \Phi\Psi)^{-1}\Phi.$$

Proof. One just has to check:

$$
\begin{aligned}
(I - \Psi\Phi)(I + \Psi(I - \Phi\Psi)^{-1}\Phi) &= I - \Psi\Phi + \Psi(I - \Phi\Psi)^{-1}\Phi\\
&\quad - \Psi\Phi\Psi(I - \Phi\Psi)^{-1}\Phi\\
&= I - \Psi\Phi + \Psi(I - \Phi\Psi)(I - \Phi\Psi)^{-1}\Phi = I,
\end{aligned}
$$

with a similar calculation for the left composition. □

The lemma is applied with

$$
\begin{aligned}
\Phi &= K_2 : H^{s+\frac{1}{2},\mu}(E_1' \oplus E_1') \to H^{s,\mu}(E),\\
\Psi &= S_{\mathscr{B}} T_0 : H^{s,\mu}(E) \to H^{s+\frac{1}{2},\mu}(E_1' \oplus E_1'),
\end{aligned}
$$

$s \geqq 0$. This replaces the construction of the inverse of $I - \Phi\Psi = I - G_2$ by the construction of the inverse of $I - \Psi\Phi = I - S_{\mathscr{B}}T_0K_2$; so that

$$\begin{aligned}(I - G_2)^{-1} &= I - K_2(I - S_1)^{-1}S_{\mathscr{B}}T_0, \text{ with}\\ S_1 &= S_{\mathscr{B}}T_0K_2.\end{aligned} \tag{3.27}$$

The advantage of this reduction is that $I - S_1$ is a pseudodifferential operator on the boundaryless manifold X'. Application of Lemma A.1 (iv) to $S' = T_0K_2$, and of Theorem 1.18 3° to $S_{\mathscr{B}}$ and S', shows that S_1 is a weakly polyhomogeneous ψdo of order -1, with symbol in $S^{-1,0} \cap S^{0,-1}$.

Now we shall use Proposition 3.6 to show:

Theorem 3.8. *The operator $S_1 = S_{\mathscr{B}}T_0K_2$ is continuous from $H^{s+\frac{1}{2},\mu}(E_1' \oplus E_1')$ to $H^{s+\frac{3}{2},\mu}(E_1' \oplus E_1')$, and satisfies*

$$\|S_1\|_{\mathscr{L}(H^{\frac{1}{2},\mu}(E_1'\oplus E_1'))} = O(\mu^{-1}), \tag{3.28}$$

on the rays in $\pm\Gamma_0$. For each closed subsector Γ of $\pm\Gamma_0$ take r_Γ so that the norm is $\leqq \frac{1}{2}$ for $|\mu| \geqq r_\Gamma$, $\mu \in \Gamma$. Then the series $S_2 = \sum_{j=1}^{\infty} S_1^j$, defining $(I - S_1)^{-1} = I + S_2$, converges in $\mathscr{L}(H^{\frac{1}{2},\mu}(E_1' \oplus E_1'))$.

The operator S_2 is a wphg ψdo with symbol in $S^{-1,0} \cap S^{0,-1}$, and $\partial_\mu^m S_2$ is a wphg ψdo with symbol in $S^{-m-1,0} \cap S^{0,-m-1}$, for each $m \in \mathbb{N}$.

Proof. The continuity of S_1 follows from (3.25), and then (3.28) holds in view of (A.7), so $S_2 = \sum_{j\geqq 1} S_1^j$ is defined as a bounded operator in $H^{\frac{1}{2},\mu}(E_1' \oplus E_1')$ for $|\mu| \geqq r_\Gamma$. The derivatives

$$\partial_\mu^m S_1 = \partial_\mu^m(S_{\mathscr{B}}T_0K_2) = \sum_{m_1+m_2+m_3=m} c_{m_1,m_2,m_3}(\partial_\mu^{m_1}S_{\mathscr{B}}\,\partial_\mu^{m_2}T_0\varphi\partial_\mu^{m_3}K_2)$$

have symbols in $S^{-m-1,0} \cap S^{0,-m-1}$ by Lemma 3.4, Proposition 3.5 and the rules of calculus, and similarly, $\partial_\mu^m S_1^j$ has symbol in $S^{-m-j-1,0} \cap S^{0,-m-j-1}$. For each m, the series $\partial_\mu^m S_2 = \sum_{j=1}^{\infty} \partial_\mu^m S_1^j$ converges in operator norm, since

$$\|\partial_\mu^m S_1^j\|_{\mathscr{L}(H^{\frac{1}{2},\mu})} \leqq C\|S_1\|^{j-m}_{\mathscr{L}(H^{\frac{1}{2},\mu})} \leqq C'2^{-j}, \text{ for } j \geqq m,\ |\mu| \geqq r_\Gamma,$$

by the Leibniz formula. To see that $S_2 \in OP'(S^{-1,0} \cap S^{0,-1})$, we note that since $I + S_2$ for each fixed μ is the inverse of $I - S_1$, it follows from the standard calculus of ψdo's that $I + S_2$ is a ψdo of order 0; and since $I - S_1$ has estimates uniform in μ (in the considered truncated sector), so does $I + S_2$, so its symbol satisfies the first requirement for belonging to $S^{0,0}$. The Leibniz formulas for $l \geqq 1$,

$$\begin{aligned}0 = \partial_z^l((I - S_1)(I + S_2)) &= \sum_{k\leqq l} \binom{l}{k} \partial_z^{l-k}(I - S_1)\partial_z^k(I + S_2)\\ &= (I - S_1)\partial_z^l S_2 - \sum_{k<l} \binom{l}{k} \partial_z^{l-k}S_1\,\partial_z^k(I + S_2),\end{aligned}$$

$$\text{imply } \partial_z^l S_2 = (I + S_2)\sum_{k<l} \binom{l}{k} \partial_z^{l-k}S_1\,\partial_z^k(I + S_2),$$

from which we conclude successively that $\partial_z^m S_2$ has symbol in S^m with uniform estimates. Thus $I + S_2$ belongs to $\mathrm{OP}'(S^{0,0})$. Then since $S_2 = S_1(I + S_2)$, it is in $\mathrm{OP}'(S^{-1,0} \cap S^{0,-1})\mathrm{OP}'(S^{0,0}) \subset OP'(S^{-1,0} \cap S^{0,-1})$. For the derivatives we can now use the above formulas (with μ instead of z) to conclude successively that $\partial_\mu^m S_2 \in OP'(S^{-1-m,0} \cap S^{0,-1-m})$ for $m > 0$. □

Inserting the formula for $(I - S_1)^{-1}$ in (3.27), we have in view of (3.18), (3.20):

$$
\begin{aligned}
\mathscr{R} &= (Q_+ - K_1 S_{\mathscr{B}} T_0)\left(I + K_2 \sum_{j=0}^{\infty} S_1^j S_{\mathscr{B}} T_0\right) \\
&= Q_+ - K_1 S_{\mathscr{B}} T_0 + Q_+ K_2 \sum_{j=0}^{\infty} S_1^j S_{\mathscr{B}} T_0 - K_1 S_{\mathscr{B}} T_0 K_2 \sum_{j=0}^{\infty} S_1^j S_{\mathscr{B}} T_0 \\
&= Q_+ - (K_1 - Q_+ K_2)(I + S_2) S_{\mathscr{B}} T_0 \\
&= Q_+ - (K_1 - K_3)(I + S_2) S_{\mathscr{B}} T_0, \quad \text{with } K_3 = Q_+ K_2.
\end{aligned}
\tag{3.29}
$$

K_3 is an sphg Poisson operator of degree -2 by Lemma A.1 (vi), and $(1 + S_2)S_{\mathscr{B}}$ is a wphg ψdo of degree 0. In view of the properties of S_2 shown in Theorem 3.8 we have:

Theorem 3.9. *For each closed subsector Γ of $\pm\Gamma_0$ one can find $r_\Gamma > 0$ so that the resolvent $\mathscr{R} = (\mathscr{P}_{\mathscr{B}} + \mu)^{-1}$ for $\mu \in \Gamma$ with $|\mu| \geqq r_\Gamma$ is of the form*

$$\mathscr{R} = Q_+ + KST_0, \tag{3.30}$$

where K resp. T_0 are a strongly polyhomogeneous Poisson resp. trace operator of degree -1 (with $\partial_\mu^m K$ and $\partial_\mu^m T_0$ sphg of degree $-1 - m$ for all m), and S is a weakly polyhomogeneous ψdo on X', with $\partial_\mu^m S \in OP'(S^{-m,0} \cap S^{0,-m})$ and wphg for all m.

In detail, $\mathscr{R}$ is described by (3.29), *with entries defined in* (3.16–20) *and Theorem 3.8.*

The operators $\partial_\mu^m \mathscr{R} : L_2(E) \to H^{m+1,\mu}(E)$ are bounded, uniformly in $\mu \in \Gamma$ for $|\mu| \geqq r_\Gamma$.

Remark 3.10. We could also have described $\mathscr{R}$ by use of the Calderón construction. Assume that Q is an inverse of $\mathscr{P} + \mu$ (for large μ in the sector) defined on a compact n-dimensional manifold $\widetilde{X}$ without boundary, in which X is smoothly imbedded (e.g. the double of the manifold extended by a cylindrical piece). Set

$$K^+ = -r^+ Q \widetilde{\gamma}_0^* \begin{pmatrix} 0 & \sigma^* \\ \sigma & 0 \end{pmatrix}, \quad P^+ = \gamma_0 K^+;$$

here P^+ is the Calderón projector for $\mathscr{P} + \mu$, and K^+ is the associated Poisson operator (cf. [S2], [H1], [G3]). In view of the invertibility of $\mathcal{P}_{\mathcal{B}} + \mu$, $\mathcal{B}P^+$ has a right inverse $\mathcal{S}_{\mathcal{B}}$, i.e.

$$\mathcal{B}P^+\mathcal{S}_{\mathcal{B}} = I.$$

Then the Poisson operator solving $(\mathcal{P} + \mu)u = 0$, $\mathcal{B}\gamma_0 u = \varphi$, equals $K_{\mathcal{B}} = K^+\mathcal{S}_{\mathcal{B}}$, and the resolvent equals

$$(\mathcal{P}_{\mathcal{B}} + \mu)^{-1} = Q_+ - K_{\mathcal{B}}\mathcal{B}\gamma_0 Q_+ = Q_+ - K^+\mathcal{S}_{\mathcal{B}}\mathcal{B}T_0.$$

Here K^+ and T_0 are an sphg Poisson resp. trace operator, and $\mathcal{S}_{\mathcal{B}}$ can be constructed such that $\mathcal{S}_{\mathcal{B}}\mathcal{B}$ has properties like those of S in Theorem 3.9. The proof is related to the preceding development, which however takes more direct advantage of the comparison of $\mathcal{P}$ with $\mathcal{P}^0$ near X'.

3.4. *Trace calculations*

Consider $\mathcal{R} = (\mathcal{P}_{\mathcal{B}} + \mu)^{-1}$, as described in Theorem 3.9. Since the injection of $H^s(X)$ into $L_2(X)$ is trace class for $s > n$, the terms in $\partial_\mu^m \mathcal{R}$ are trace class when $m \geqq n$.

Theorem 3.11. *Let φ be any morphism in $E = E_1 \oplus E_2$, and let $m \geqq n = \dim X$. Then*

$$\text{(3.31)}\quad \operatorname{Tr}(\varphi\partial_\mu^m(\mathcal{P}_{\mathcal{B}} + \mu)^{-1}) \sim a_0\mu^{n-m-1} + \sum_{j=1}^{\infty}(a_j + b_j)\mu^{n-m-1-j} + \sum_{j=0}^{\infty}(c_j \log\mu + c_j')\mu^{-m-1-j},\ \textit{as } |\mu| \to \infty,$$

for μ in closed subsectors of $\pm\Gamma_0$. The coefficients a_j, b_j and c_j are integrals, $\int_X a_j(x)\,dx$, $\int_{X'} b_j(x')\,dx'$ and $\int_{X'} c_j(x')dx'$, of densities locally determined by the symbols of P and B, while the c_j' are in general globally determined. The coefficients c_0 and c_0' are the same as for the case where, in (3.1) *and* (3.2), *the P_j and P_j' are zero* (*the "cylindrical" case*).

Proof. We find from (3.29):

$$\varphi\partial_\mu^m(\mathcal{P}_{\mathcal{B}} + \mu)^{-1} = \varphi\partial_\mu^m Q_+ - \varphi\partial_\mu^m[K_1 S_{\mathcal{B}} T_0] - \varphi\partial_\mu^m[(K_1 S_2 - K_3(I + S_2))S_{\mathcal{B}}T_0].$$

First, $\operatorname{Tr}(\varphi\partial_\mu^m Q_+)$ contributes the well-known expansion $\sum_0^\infty a_j\mu^{n-m-1-j}$.

For the other terms we can use the invariance of the trace under cyclic permutation of the operators, to reduce to a study of operators on X'. For the middle term we have:

$$\begin{aligned}\operatorname{Tr}(\varphi\partial_\mu^m[K_1 S_{\mathcal{B}} T_0]) &= \sum_{m_1+m_2+m_3=m} c_{m_1,m_2,m_3}\operatorname{Tr}(\varphi\partial_\mu^{m_1}K_1\,\partial_\mu^{m_2}S_{\mathcal{B}}\,\partial_\mu^{m_3}T_0)\\ &= \sum_{m_1+m_2+m_3=m} c_{m_1,m_2,m_3}\operatorname{Tr}(\partial_\mu^{m_2}S_{\mathcal{B}}\,\partial_\mu^{m_3}T_0\varphi\partial_\mu^{m_1}K_1),\end{aligned}$$

where the expressions $\partial_\mu^{m_3} T_0 \varphi \partial_\mu^{m_1} K_1$ are sphg ψdo's on X' of degree $-m_3 - m_1 - 1$, by Lemma 3.4. Then by Theorem 1.18 and Proposition 3.5, $\partial_\mu^{m_2} S_{\mathscr{B}} \partial_\mu^{m_3} T_0 \varphi \partial_\mu^{m_1} K_1$ is a weakly polyhomogeneous ψdo on X' of degree $-m - 1$, with symbol in $S^{-m-1,0} \cap S^{0,-m-1}$.

To this we can apply our general Theorem 2.1, after a reduction to local trivializations by use of a partition of unity. Since the symbol has degrees $-m - 1 - j$, $j \geqq 0$, and μ-exponent $d = -m - 1$, we get an expansion in a series of locally determined terms $b_{k,1}\mu^{-m-1+(n-1)-k}$, $k \geqq 0$, together with a series of terms $(c_{j,1} \log \mu + c'_{j,1})\mu^{-m-1-j}$, $j \geqq 0$, with $c_{j,1}$ locally determined.

The third term is treated similarly; here

$$\operatorname{Tr}(\partial_\mu^m[(K_1 S_2 - K_3(I + S_2))S_{\mathscr{B}} T_0]) = \operatorname{Tr}(\partial_\mu^m[S_2 S_{\mathscr{B}} T_0 K_1 - (I + S_2)S_{\mathscr{B}} T_0 K_3]),$$

with related formulas when a nontrivial φ enters; and we use that $\partial_\mu^{m_3} T_0 \varphi \partial_\mu^{m_1} K_1$ and $\partial_\mu^{m_3} T_0 \varphi \partial_\mu^{m_1} K_3$ are sphg ψdo's of degree $-m_3 - m_1 - 1$ resp. $-m_3 - m_1 - 2$, together with the information on the weakly polyhomogeneous factors. We find that the operator whose trace is to be calculated has symbol in $S^{-m-2,0} \cap S^{0,-m-2}$, polyhomogeneous with degrees $-m - 2 - j$, $j \geqq 0$, and μ-exponent $d = -m - 2$, so Theorem 2.1 here gives an expansion in a series of locally determined terms $b_{k,2}\mu^{-m-2+(n-1)-k}$, $k \geqq 0$, together with a series of terms $(c_{j,2} \log \mu + c'_{j,2})\mu^{-m-1-j}$, $j \geqq 1$, with $c_{j,2}$ locally determined.

Taken together, this gives the expansion (3.31).

Now observe that the terms $(c_0 \log \mu + c'_0)\mu^{-m-1}$ in (3.31) come only from $\operatorname{Tr}(\varphi \partial_\mu^m [K_1 \mathscr{S}_{\mathscr{B}} T_0])$. This implies the last statement in Theorem 3.11. For, K_1 and $S_{\mathscr{B}}$ are the same as for the case where the P_j and P'_j are 0. The third factor $T_0 = \gamma_0 Q_+$ uses the symbol of $(\mathscr{P} + \mu)^{-1}$ evaluated at $x_n = 0$. The leading term of this is the same as for the case where P_j and P'_j are 0, and the lower order terms contribute ultimately an operator in $\mathrm{OP}'(S^{-m-2,0} \cap S^{0,-m-2})$. The first possible nonlocal and log contributions from this are the terms with μ^{-m-2} and $\mu^{-m-2} \log \mu$. □

It is easy to draw conclusions from this on asymptotic expansions for $\varphi(P_B{}^* P_B - \lambda)^{-1}$ and $\varphi P_B(P_B{}^* P_B - \lambda)^{-1}$ etc., in view of (3.9).

Corollary 3.12. *Let $\varphi_{kl} : E_l \to E_k$ be morphisms, for $k, l = 1, 2$.*

The traces $\operatorname{Tr}(\varphi_{11}\partial_\lambda^m(P_B{}^* P_B - \lambda)^{-1})$ *and* $\operatorname{Tr}(\varphi_{22}\partial_\lambda^m(P_B P_B{}^* - \lambda)^{-1})$ *have asymptotic expansions (for $k = 1$ resp. 2):*

$$(3.32) \qquad a_{0,kk}(-\lambda)^{\frac{n}{2}-m-1} + \sum_{j=1}^{\infty}(a_{j,kk} + b_{j,kk})(-\lambda)^{\frac{n-j}{2}-m-1} + \sum_{j=0}^{\infty}(c_{j,kk} \log \lambda + c'_{j,kk})(-\lambda)^{\frac{-j}{2}-m-1};$$

and $\mathrm{Tr}(\varphi_{12}\partial_\lambda^m P_B(P_B{}^*P_B-\lambda)^{-1})$ *and* $\mathrm{Tr}(\varphi_{21}\partial_\lambda^m P_B{}^*(P_BP_B{}^*-\lambda)^{-1})$ *have asymptotic expansions (for* $\{k,l\}=\{1,2\}$ *resp.* $\{2,1\}$*):*

$$(3.33)\qquad a_{0,kl}(-\lambda)^{\frac{n-1}{2}-m}+\sum_{j=1}^{\infty}(a_{j,kl}+b_{j,kl})(-\lambda)^{\frac{n-j-1}{2}-m}$$
$$+\sum_{j=0}^{\infty}(c_{j,kl}\log\lambda+c'_{j,kl})(-\lambda)^{\frac{-j-1}{2}-m}$$

with coefficients described as in Theorem 3.11.

The coefficients $c_{0,kl}$ *and* $c'_{0,kl}$ *are the same as those for the cylindrical case.*

Proof. Using (3.9), take

$$\varphi=\begin{pmatrix}\varphi_{11}&0\\0&0\end{pmatrix},\quad\begin{pmatrix}0&0\\0&\varphi_{22}\end{pmatrix},\quad\begin{pmatrix}0&\varphi_{12}\\0&0\end{pmatrix},\ \text{resp.}\ \begin{pmatrix}0&0\\\varphi_{21}&0\end{pmatrix},$$

in Theorem 3.11, and divide by μ in the first two cases. Now replace μ by $(-\lambda)^{\frac{1}{2}}$ and note that $\partial_\lambda=(2\mu)^{-1}\partial_\mu$. □

These results yield asymptotic expansions of the traces of heat operators $\varphi_{11}e^{-tP_B{}^*P_B}$, $\varphi_{12}P_Be^{-tP_B{}^*P_B}$, etc., and power operators $\varphi_{11}(P_B{}^*P_B)^{-s}=:\varphi_{11}Z(P_B^*P_B,s)$, $\varphi_{12}P_B(P_B{}^*P_B)^{-s}$, etc., by use of the transition formulas in [GS1]:

Theorem 3.13. *There are coefficients* $\tilde a_{j,kl}$, $\tilde b_{j,kl}$, $\tilde c_{j,kl}$, $\tilde c'_{j,kl}$, *related by suitable gamma factors to those in Corollary 3.12 (see the proof below for further details) such that, with* $v_1=\mathrm{Tr}(\varphi_{11}\Pi_0(P_B))$, $v_2=\mathrm{Tr}(\varphi_{22}\Pi_0(P_B{}^*))$, *the zeta and eta functions have singularity structures described by:*

$$(3.34)\qquad \Gamma(s)\mathrm{Tr}(\varphi_{kk}Z(P_B{}^*P_B,s))\sim\frac{-v_k}{s}+\frac{\tilde a_{0,kk}}{s-\frac{n}{2}}+\sum_{j=1}^{\infty}\frac{\tilde a_{j,kk}+\tilde b_{j,kk}}{s-\frac{n-j}{2}}$$
$$+\sum_{j=0}^{\infty}\Big(\frac{\tilde c_{j,kk}}{(s+\frac{j}{2})^2}+\frac{\tilde c'_{j,kk}}{s+\frac{j}{2}}\Big);$$

$\Gamma(s)\mathrm{Tr}(\varphi_{12}P_BZ(P_B{}^*P_B,s))$ *resp.* $\Gamma(s)\mathrm{Tr}(\varphi_{21}P_B{}^*Z(P_BP_B{}^*,s))$

$$\sim\frac{\tilde a_{0,kl}}{s-\frac{n+1}{2}}+\sum_{j=1}^{\infty}\frac{\tilde a_{j,kl}+\tilde b_{j,kl}}{s-\frac{n-j+1}{2}}+\sum_{j=0}^{\infty}\Big(\frac{\tilde c_{j,kl}}{(s+\frac{j-1}{2})^2}+\frac{\tilde c'_{j,kl}}{s+\frac{j-1}{2}}\Big).$$

The heat traces have the asymptotic behavior for $t\to0$*:*

$$(3.35)\quad \mathrm{Tr}(\varphi_{kk}e^{-tP_B{}^*P_B})\sim\tilde a_{0,kk}t^{-\frac{n}{2}}+\sum_{j=1}^{\infty}\big(\tilde a_{j,kk}+\tilde b_{j,kk}\big)t^{\frac{j-n}{2}}$$
$$+\sum_{j=0}^{\infty}\big(-\tilde c_{j,kk}t^{\frac{j}{2}}\log t+\tilde c'_{j,kk}t^{\frac{j}{2}}\big),$$

$\mathrm{Tr}(\varphi_{12}P_Be^{-tP_B{}^*P_B})$ *resp.* $\mathrm{Tr}(\varphi_{21}P_B{}^*e^{-tP_BP_B{}^*})$

$$\sim\tilde a_{0,kl}t^{-\frac{n+1}{2}}+\sum_{j=1}^{\infty}\big(\tilde a_{j,kl}+\tilde b_{j,kl}\big)t^{\frac{j-n-1}{2}}+\sum_{j=0}^{\infty}\big(-\tilde c_{j,kl}t^{\frac{j-1}{2}}\log t+\tilde c'_{j,kl}t^{\frac{j-1}{2}}\big).$$

The $\tilde{c}'_{j,kl}$ and v_k are in general globally defined, while the other coefficients are local. The coefficients $\tilde{c}_{0,kl}$ and $\tilde{c}'_{0,kl}$ are the same as those for the cylindrical case.

Proof. Recall that the resolvent, power function and exponential function of a selfadjoint operator $S \geqq 0$ with compact resolvent are related to one another by the formulas

$$\begin{aligned}(3.36)\qquad Z(S,s) = S^{-s} &= \tfrac{i}{2\pi}\int_C \lambda^{-s}(S-\lambda)^{-1}d\lambda \\ &= \tfrac{1}{(s-1)\cdots(s-m)}\tfrac{i}{2\pi}\int_C \lambda^{m-s}\partial_\lambda^m(S-\lambda)^{-1}\,d\lambda, \\ e^{-tS}(I-\Pi_0(S)) &= \tfrac{i}{2\pi}\int_C e^{-t\lambda}(S-\lambda)^{-1}\,d\lambda \\ &= \tfrac{1}{2\pi i}\int_{\operatorname{Re}s=c} t^{-s}Z(S,s)\Gamma(s)\,ds, \quad c>0;\end{aligned}$$

here S^{-s} is taken to be zero on the nullspace of S, and C is a curve

$$\begin{aligned}C_{\theta,r_0} &= \{\,\lambda = re^{i\theta} \mid \infty > r \geqq r_0\,\} + \{\,\lambda = r_0 e^{i\theta'} \mid \theta \geqq \theta' \geqq -\theta\,\} \\ &\quad + \{\,\lambda = re^{i(2\pi-\theta)} \mid r_0 \leqq r < \infty\,\},\end{aligned}$$

with $0 < \theta \leqq \pi$, and r_0 chosen so that $(S-\lambda)^{-1}$ is holomorphic on $0 < |\lambda| \leqq r_0$. In the second formula, one must take $\theta < \frac{\pi}{2}$. We apply this to $S = P_B{}^*P_B$ and $P_BP_B{}^*$ and compose with a differential operator D; by taking traces we then get similar formulas relating the traces of $D\partial_\lambda^m(S-\lambda)^{-1}$, $DZ(S,s)$ and De^{-tS}, provided that $\operatorname{Re}s$ and m are taken large enough.

To obtain the first expansion in (3.34), let $r(\lambda) = \operatorname{Tr}(\varphi_{11}\partial_\lambda^m(P_B{}^*P_B-\lambda)^{-1})$, and note that besides having the asymptotic expansion (3.32) for $\lambda \to \infty$ in $S_\delta = \{|\pi - \arg\lambda| < \delta\}$, any $\delta < \pi$, $r(\lambda)$ is meromorphic at 0 with

$$r(\lambda) = v_1 m!(-\lambda)^{-m-1} + r_0(\lambda), \quad v_1 = \operatorname{Tr}(\varphi_{11}\Pi_0(P_B)), \quad r_0(\lambda) \text{ holomorphic at } 0.$$

Let $\zeta(s) = \operatorname{Tr}(\varphi_{11}Z(P_B{}^*P_B,s))$. By [GS1, Prop. 2.9], relating the zeta poles to the asymptotic properties of the resolvent, it follows that the function

$$\psi(s) = \frac{\pi\zeta(s)(s-1)\cdots(s-m)}{\sin\pi s} = \frac{\pi}{\sin\pi s}\frac{i}{2\pi}\int_C \lambda^{m-s}\partial_\lambda^m(P_B{}^*P_B-\lambda)^{-1}\,d\lambda$$

has the singularity structure

$$\begin{aligned}\psi(s) \sim -\frac{v_1 m!}{s} - \sum_{j\geqq 0}\frac{b_j}{s-m-j-1} + \frac{a_{0,11}}{s-\frac{n}{2}} \\ + \sum_{j=1}^{\infty}\frac{a_{j,11}+b_{j,11}}{s-\frac{n-j}{2}} + \sum_{j=0}^{\infty}\Big(\frac{c_{j,11}}{(s+\frac{j}{2})^2} + \frac{c'_{j,11}}{s+\frac{j}{2}}\Big);\end{aligned}$$

here the b_j are the coefficients in the power series $r_0(\lambda) = \sum_{j\geq 0} b_j(-\lambda)^j$ (convergent near 0). Since $\pi(\sin \pi s)^{-1} = \Gamma(s)\Gamma(1-s)$,

$$\Gamma(s)\zeta(s) = \frac{\psi(s)}{(s-1)\cdots(s-m)\Gamma(1-s)} = \frac{\psi(s)}{(-1)^m\Gamma(m+1-s)}.$$

We can assume that m is even. The poles of $\Gamma(m+1-s)$ cancel the poles in the sum $\sum_{j\geq 0}\frac{b_j}{s-m-j-1}$, so we are left with the singularities

$$\begin{aligned}(3.37)\quad \Gamma(s)\zeta(s) \sim &-\frac{v_1}{s} + \frac{a_{0,11}}{\Gamma(m+1+\frac{n}{2})(s-\frac{n}{2})}\\ &+\sum_{j=1}^{\infty}\frac{a_{j,11}+b_{j,11}}{\Gamma(m+1+\frac{n-j}{2})(s-\frac{n-j}{2})}\\ &+\sum_{j=0}^{\infty}\Big(\frac{c_{j,11}}{\Gamma(m+1-\frac{j}{2})(s+\frac{j}{2})^2}+\frac{c'_{j,11}}{\Gamma(m+1-\frac{j}{2})(s+\frac{j}{2})}\Big).\end{aligned}$$

This shows the first expansion in (3.34), for $k=1$. The first expansion in (3.35) is obtained from (3.37) by use of [GS1, Prop. 5.1], that relates the zeta poles to the asymptotics of the heat trace at zero. This gives:

$$\begin{aligned}(3.38)\qquad \operatorname{Tr}(\varphi_{11}e^{-tP_B{}^*P_B}(I-\Pi_0(P_B))) &\sim -v_1 + \tfrac{a_{0,11}}{\Gamma(m+1+\frac{n}{2})}t^{-\frac{n}{2}}\\ +\sum_{j=1}^{\infty}\tfrac{a_{j,11}+b_{j,11}}{\Gamma(m+1+\frac{n-j}{2})}t^{\frac{j-n}{2}} &+ \sum_{j=0}^{\infty}\Big(-\tfrac{c_{j,11}}{\Gamma(m+1-\frac{j}{2})}t^{\frac{j}{2}}\log t+\tfrac{c'_{j,11}}{\Gamma(m+1-\frac{j}{2})}t^{\frac{j}{2}}\Big).\end{aligned}$$

In view of the definition of v_1, this shows the first expansion in (3.35) for $k=1$.

The other expansions are obtained in a similar way, using also (3.33). □

For the cylindrical case, the coefficients were determined exactly (in terms of the zeta and eta functions of A) in [GS1], for cases where B_0 is a projection in V_0, cf. Assumption 3.1. Thus we find from [GS1, Cor. 2.7–2.8], in the case $B_0 = \Pi_0(A)$:

$$\begin{aligned}&\textbf{For } n \textbf{ even, } \tilde{c}_{0,kk} = (-1)^k\frac{c_{n-1}(\varphi^0_{kk}A,A^2)}{4\sqrt{\pi}},\\ (3.39)\qquad &\tilde{c}'_{0,kk} = (-1)^k\Big[\frac{c'_{n-1}(\varphi^0_{kk}A,A^2)}{4\sqrt{\pi}}+\tfrac{1}{4}\operatorname{Tr}(\varphi^0_{kk}\Pi_0(A))\Big].\\ &\textbf{For } n \textbf{ odd, } \tilde{c}_{0,kk} = 0,\quad \tilde{c}'_{0,kk} = (-1)^k\tfrac{1}{4}\big[\eta(\varphi^0_{kk},A,0)+\operatorname{Tr}(\varphi^0_{kk}\Pi_0(A))\big].\end{aligned}$$

Here φ^0_{kk} is the restriction of φ_{kk} to ∂X, and $\frac{1}{s}c_{n-1}(\varphi^0_{kk}A,A^2)+c'_{n-1}(\varphi^0_{kk}A,A^2)$ are the first Laurent terms of $\Gamma(s+\frac{1}{2})\eta(\varphi^0_{kk},A,2s)$ at $s=0$. (This extends the result of [G2] for the case $\varphi_{kk}=I$, where $c_{n-1}=0$ and $c'_{n-1}=\sqrt{\pi}\,\eta(A,0)$.) Similarly, we get from [GS1, Cor. 4.3], for the singularity of $\operatorname{Tr}(\varphi_{12}P_BZ(P_B{}^*P_B,s))$ at $s=\frac{1}{2}$ ($B_0=\Pi_0(A)$):

For n even, $\tilde{c}_{0,12} = \frac{1}{4\pi} c_{n-1}(\varphi_{12}^0 \sigma A, A^2)$,

(3.40) $$\tilde{c}'_{0,12} = \tfrac{1}{4\pi} c'_{n-1}(\varphi_{12}^0 \sigma A, A^2) + \tfrac{1}{4\sqrt{\pi}} \operatorname{Tr}(\varphi_{12}^0 \sigma \Pi_0(A)).$$

For n odd, $\tilde{c}_{0,12} = 0, \quad \tilde{c}'_{0,12} = \frac{1}{4\sqrt{\pi}} \left[\eta(\varphi_{12}^0 \sigma, A, 0) + \operatorname{Tr}(\varphi_{12}^0 \sigma \Pi_0(A)) \right].$

Similar formulas hold for the singularity of $\operatorname{Tr}(\varphi_{21} P_B{}^* Z(P_B P_B{}^*, s))$, just with $\varphi_{12}^0 \sigma$ replaced by $\sigma^* \varphi_{21}^0$.

Special results for some other choices of B_0 are likewise easily inferred from [GS1].

The considerations in [GS1, Sect. 4.3] on the variation at 0 of the eta function of P_B in certain selfadjoint cases can be generalized to the present situation.

Remark 3.14. Similar considerations allow the calculation of $\operatorname{Tr}(D \partial_\mu^m \mathscr{R})$ when D is an arbitrary differential operator on X, for $m \geqq n + d$, $d =$ the order of D. One finds that

(3.41) $$\begin{aligned} \operatorname{Tr}(D \partial_\mu^m \mathscr{R}) \sim\ & a_0(D,P) \mu^{n-m+d-1} \\ & + \sum_{j=1}^{\infty} (a_j(D,P) + b_j(D,P_B)) \mu^{n-m+d-1-j} \\ & + \sum_{j=0}^{\infty} (c_j(D,P_B) \log \mu + c'_j(D,P_B)) \mu^{-m+d-1-j} \end{aligned}$$

(the primed coefficients global, the others local); and consequences are drawn as above for the corresponding zeta and eta functions and exponential traces.

Appendix

The composition rules for operators as in (3.21)–(3.22) with strongly polyhomogeneous symbol-kernels (see Definition 3.3 ff.), taken together with differential operators and resolvent parametrices, follow directly from the composition rules for "classical polyhomogeneous" operators in one more variable, treated in [BM], [G1]. To make the present paper reasonably selfcontained, we show the rules in the μ-dependent framework in some detail below.

Weakly polyhomogeneous boundary operators can also be constructed, but with more complicated composition rules, so since they are not needed for the present purposes, they are not included here.

Lemma A.1. *Suppose that:*

$Q = OP(q(x,\xi,\mu))$ is the resolvent parametrix of an elliptic differential operator $A = \mathrm{OP}(a(x,\xi))$ on $\mathbb{R}^n$ of degree m_1 with $a_{m_1}(x,\xi) - \mu^{m_1}$ invertible for $\mu \in \Gamma$.

$D = \mathrm{OP}(d(x,\xi))$ is a differential operator on $\mathbb{R}^n$ of degree m_2.

$K = \mathrm{OPK}(\tilde{k}(x,\xi',\mu))$ is an sphg Poisson operator on $\overline{\mathbb{R}}_+^n$, of degree m_3.

$T = \mathrm{OPT}(\tilde{t}(x,\xi',\mu))$ is an sphg class 0 trace operator on $\overline{\mathbb{R}}_+^n$, of degree m_4.

$S = \mathrm{OP}'(s(x', \xi', \mu))$ is an sphg ψdo of degree m_5 on $\mathbb{R}^{n-1}$.

Then:

(i) *$\gamma_0 K$ is an sphg ψdo of degree $m_3 + 1$ on $\mathbb{R}^{n-1}$, with symbol $\tilde{k}(x', 0, \xi', \mu)$.*

(ii) *$\gamma_0 Q_+$ is an sphg trace operator of degree $-m_1$ and class 0, with symbol-kernel $r^+ \mathscr{F}^{-1}_{\xi_n \to -y_n} q(x', 0, \xi, \mu)$.*

(iii) *$r^+ DQ\tilde{\gamma}_0^*$ is an sphg Poisson operator of degree $m_2 - m_1$; here $\tilde{\gamma}_0^*$ denotes the adjoint of the "two-sided" trace operator $\tilde{\gamma}_0 : u \mapsto u|_{x_n=0}$ (going from $C_0^\infty(\mathbb{R}^n)$ to $C_0^\infty(\mathbb{R}^{n-1})$); it can also be written as $\tilde{\gamma}_0^* : v(x') \mapsto v(x') \otimes \delta(x_n)$. When DQ is written in (x', y_n)-form (cf. Remark 1.20), $DQ = OP(q_1(x', y_n, \xi, \mu))$, then the symbol-kernel of the Poisson operator $r^+ DQ\tilde{\gamma}_0^*$ is $r^+ \mathscr{F}^{-1}_{\xi_n \to x_n} q_1(x', 0, \xi, \mu)$.*

(iv) *TK is an sphg ψdo of degree $m_4 + m_3 + 1$ on $\mathbb{R}^{n-1}$, with symbol*

$$\int_0^\infty \tilde{t}(x', x_n, \xi', \mu) \circ' \tilde{k}(x', x_n, \xi', \mu)\, dx_n,$$

where $\circ'$ denotes symbol composition (cf. Definition 1.19) with respect to the x' variable.

(v) *$x_n^k KS$ is an sphg Poisson operator of degree $-k + m_3 + m_5$, with symbol-kernel $x_n^k \tilde{k}(x, \xi', \mu) \circ' s(x', \xi', \mu)$.*

(vi) *$Q_+ K$ is an sphg Poisson operator of degree $-m_1 + m_3$, with symbol-kernel described in the proof below.*

(vii) *∂_μ preserves strong polyhomogeneity and reduces the degree of any of the operators by 1.*

The rules extend to operators on manifolds.

Proof. As usual, the compositions may need precautions: for symbols with local estimates in x' as in [BM], [G1], one factor should be properly supported, but for globally estimated symbols the compositions are generally defined as in [GK].

The proofs of (i), (iv), (v) and (vii) are straightforward.

For (ii), let

$$\tilde{q}(x, z_n, \xi', \mu) = \mathscr{F}^{-1}_{\xi_n \to z_n} q(x, \xi, \mu). \tag{A.1}$$

(So Q has the "symbol-kernel" $\tilde{q}(x', x_n, x_n - y_n, \xi', \mu)$, when one replaces the ψdo action in the n'th variable by an integral operator action.) Then

$$\begin{aligned}[\gamma_0 Q_+ u](x') &= \gamma_0 r^+ \int_{\mathbb{R}^n} e^{ix\cdot\xi} q(x, \xi, \mu) \widehat{e^+ u}(\xi)\, đ\xi \\ &= \gamma_0 r^+ \int_{\mathbb{R}^{n-1}} e^{ix'\cdot\xi'} \int_0^\infty \tilde{q}(x', x_n, x_n - y_n, \xi', \mu) \acute{u}(\xi', y_n)\, dy_n\, đ\xi' \\ &= \int_{\mathbb{R}^{n-1}} e^{ix'\cdot\xi'} \int_0^\infty \tilde{q}(x', 0, -y_n, \xi', \mu) \acute{u}(\xi', y_n)\, dy_n\, đ\xi'.\end{aligned}$$

This shows that $\gamma_0 Q_+$ is a trace operator as in (3.22), with symbol-kernel $\tilde{t} = r^+ \tilde{q}(x', 0, -y_n, \xi', \mu)$. For this function, the properties in Definition 3.3 (with

$m = -m_1$) follow easily from the fact that q has an asymptotic expansion in homogeneous functions which are rational in (ξ, μ), with denominators that are powers of the elliptic polynomial $\det(a_{m_1} - \mu^{m_1})$.

For (iii), let

(A.2) $$\tilde{q}_1(x', y_n, z_n, \xi', \mu) = \mathscr{F}^{-1}_{\xi_n \to z_n} q_1(x', y_n, \xi, \mu);$$

then DQ has the "symbol-kernel" $\tilde{q}_1(x', y_n, x_n - y_n, \xi', \mu)$. Now

$$\begin{aligned}[r^+ Q \tilde{\gamma}_0^* v](x) &= r^+ \int_{\mathbb{R}^{2n}} e^{i(x'-y')\cdot\xi' + i(x_n - y_n)\xi_n} q_1(x', y_n, \xi, \mu)[v(y') \otimes \delta(y_n)]\, dy\, đ\xi \\ &= r^+ \int_{\mathbb{R}^n} e^{ix'\cdot\xi' + ix_n\xi_n} q_1(x', 0, \xi, \mu)\hat{v}(\xi')\, đ\xi \\ &= r^+ \int_{\mathbb{R}^{n-1}} e^{ix'\cdot\xi'} \tilde{q}_1(x', 0, x_n, \xi', \mu)\hat{v}(\xi')\, đ\xi', \text{ for } x_n > 0;\end{aligned}$$

this is a Poisson operator as in (3.21) with symbol-kernel $r^+\tilde{q}_1(x', 0, x_n, \xi', \mu)$. (The formulas must be suitably interpreted; one can use an approximation of $\delta(y_n)$ by smooth functions.) The properties of the symbol-kernel follow as in (ii).

For (vi), recall that $Q_+ = r^+ Q e^+$, cf. (3.14) ff. Write K in terms of a symbol-kernel $\tilde{k}'$ in y'-form:

$$Kv(x) = \int_{\mathbb{R}^{2n-2}} e^{i(x'-y')\cdot\xi'} \tilde{k}'(y', x_n, \xi', \mu) v(y')\, dy'\, đ\xi'$$

(by Theorem 1.18 1° applied with respect to the boundary variables), and introduce the associated *symbol* $k'(y', \xi, \mu) = \mathscr{F}_{x_n \to \xi_n} e^+ \tilde{k}'(y', x_n, \xi', \mu)$; then

(A.3) $$e^+ Kv(x) = \int_{\mathbb{R}^{2n-1}} e^{i(x'-y')\cdot\xi' + ix_n\xi_n} k'(y', \xi, \mu) v(y')\, dy'\, đ\xi.$$

Consider a Taylor expansion of $q(x, \xi, \mu)$ in x_n:

(A.4) $$q(x, \xi, \mu) = \sum_{j<N} \tfrac{1}{j!} x_n^j \partial_{x_n}^j q(x', 0, \xi, \mu) + x_n^N r_N(x, \xi, \mu).$$

For each j one finds:

$$\begin{aligned}&r^+ \mathrm{OP}(\partial_{x_n}^j q(x', 0, \xi, \mu)) e^+ Kv \\ &= r^+ \int_{\mathbb{R}^{2n}} e^{i(x-z)\cdot\xi} \partial_{x_n}^j q(x', 0, \xi, \mu)(e^+ Kv)(z)\, dz\, đ\xi \\ &= r^+ \int_{\mathbb{R}^{4n-1}} e^{i(x-z)\cdot\xi + i(z'-y')\cdot\eta' + iz_n\eta_n} \partial_{x_n}^j q(x', 0, \xi, \mu) k'(y', \eta, \mu) v(y')\, dy'\, đ\eta\, dz\, đ\xi \\ &= r^+ \int_{\mathbb{R}^{3n-1}} e^{i(x-z)\cdot\xi} \partial_{x_n}^j q(x', 0, \xi, \mu)[\mathscr{F}^{-1}_{\eta \to (z'-y', z_n)} k'(y', \eta, \mu)] v(y')\, dy'\, dz\, đ\xi \\ &= r^+ \int_{\mathbb{R}^{2n-1}} e^{i(x'-y')\cdot\xi' + ix_n\xi_n} \partial_{x_n}^j q(x', 0, \xi, \mu) k'(y', \xi, \mu) v(y')\, dy'\, đ\xi \\ &= K_j v,\end{aligned}$$

where K_j is the Poisson operator with symbol-kernel in (x', y')-form:

$$\text{(A.5)}\qquad \tilde{k}_j(x', y', x_n, \xi', \mu) = r^+ \mathscr{F}^{-1}_{\xi_n \to x_n}[\partial^j_{x_n} q(x', 0, \xi, \mu) k'(y', \xi, \mu)].$$

One can show that the remainder in (A.4) gives an operator with order going to $-\infty$ in the appropriate sense for $N \to \infty$, so it is altogether found that the composed operator Q_+K is a Poisson operator with a symbol-kernel $\tilde{h}$ in (x', y')-form described by

$$\text{(A.6)}\qquad \tilde{h}(x', y', x_n, \xi', \mu) \sim \sum_{j \geqq 0} \tfrac{1}{j!} x_n^j \tilde{k}_j(x', y', x_n, \xi', \mu).$$

This is reduced to a symbol-kernel in x'-form by Theorem 1.18 3°.

The strong poly(quasi)homogeneity property (Definition 3.3) of $\tilde{k}$ carries over to a similar property of $\tilde{k}'$, which is reflected in a suitable polyhomogeneity property of k'. Multiplication by the symbols $\partial^j_{x_n} q$ (that are series of rational functions as described under (ii)) preserves this kind of polyhomogeneity, which then translates back to a strong poly(quasi)homogeneity property of the $\tilde{k}_j$ in (A.5); and it is found altogether that the symbol-kernel of Q_+K is as in Definition 3.3 with $m = m_3 - m_1$. □

More details can be found in [G1], [GK].

Example A.2. A simple example of an sphg symbol-kernel of degree -1 is $\tilde{p} = e^{-x_n\langle\xi', \mu\rangle}$, with principal part $\tilde{p}_{-1} = e^{-x_n[\xi', \mu]}$ (cf. (1.1)), and with associated symbol $p = (\langle\xi', \mu\rangle + i\xi_n)^{-1}$, whose principal part $([\xi', \mu] + i\xi_n)^{-1}$ is clearly homogeneous of degree -1 in (ξ', ξ_n, μ) for $|\xi'| \geqq 1$. To illustrate (ii) and (iii) above, we observe that for the symbol $q(\xi, \mu)$ of $\mathscr{Q} = (1 - \Delta + \mu^2)^{-1}$,

$$q(\xi, \mu) = \frac{1}{1 + |\xi|^2 + \mu^2} = \frac{1}{2\langle\xi', \mu\rangle}\left(\frac{1}{\langle\xi', \mu\rangle + i\xi_n} + \frac{1}{\langle\xi', \mu\rangle - i\xi_n}\right);$$

its inverse Fourier transform in ξ_n is $\tilde{q}(x_n, \xi', \mu) = \frac{1}{2\langle\xi', \mu\rangle} e^{-|x_n|\langle\xi', \mu\rangle}$, so $r^+q(-x_n, \xi', \mu)$ and $r^+q(x_n, \xi', \mu)$ are both equal to $\frac{1}{2\langle\xi', \mu\rangle} e^{-x_n\langle\xi, \mu\rangle}$.

The construction in (iii) is in a certain sense dual to the construction in (ii) (when $D = I$). This kind of Poisson operator entered in a crucial way already in the introductions of ψdo methods into the theory of differential elliptic boundary problems (defining the Calderón projector), in Seeley [S2, Th. 5] and Hörmander [H1, Lemma 2.1.3].

One could allow a factor D in (ii), but it will generally give rise to trace operators of class > 0, that we do not need to deal with systematically here. Q can be replaced by more general ψdo's satisfying the transmission condition, cf. [BM], [G1], [GK]. Products KT belong to the so-called singular Green operators; we shall not here need the general type G entering in the full calculus.

The sphg operators are "of regularity $+\infty$" in the terminology of [G1], [GK]. The mapping properties of these operators are covered by the general rules of [G1, Sect. 2.5], [GK, Sect. 4], but for completeness we shall include elementary proofs for the cases used here. Introduce the parameter-dependent normed Sobolev spaces

$$H^{s,\mu}(\mathbb{R}^n) = \{ v \in S'(\mathbb{R}^n) \mid \langle \xi, \mu \rangle^s \hat{v}(\xi) \in L_2(\mathbb{R}^n) \}, \text{ with norm } \|\langle \xi, \mu \rangle^s \hat{v}\|_{L_2},$$

$$H^{s,\mu}(\overline{\mathbb{R}}^n_+) = r^+ H^{s,\mu}(\mathbb{R}^n), \text{ with norm } \|u\|_{H^{s,\mu}(\overline{\mathbb{R}}^n_+)} = \inf\{ \|v\|_{H^{s,\mu}(\mathbb{R}^n)} \mid u = r^+ v \},$$

with the usual generalizations to vector bundles E over manifolds X (notation $H^{s,\mu}(X,E)$ or just $H^{s,\mu}(E)$); note that

$$\text{(A.7)} \qquad \langle \mu \rangle^s \|u\|_{L_2} + \|u\|_{H^s} \simeq \|u\|_{H^{s,\mu}}, \text{ for } s \geqq 0;$$

$$\langle \mu \rangle \|u\|_{H^{s,\mu}} \leqq C_s \|u\|_{H^{s+1,\mu}} \text{ for } s \in \mathbb{R}.$$

We write

$$\mathrm{OP}'\big(\langle \xi', \mu \rangle^s\big) = \langle \mathrm{D}', \mu \rangle^s.$$

Lemma A.3. *When K is a globally estimated sphg Poisson operator of degree $m \in \mathbb{Z}$, then it is bounded*

$$\text{(A.8)} \quad K : H^{s+m+\frac{1}{2},\mu}(\mathbb{R}^{n-1}) \to H^{s,\mu}(\overline{\mathbb{R}}^n_+), \quad \textit{uniformly in } \mu, \textit{ for } s \in \mathbb{R}.$$

When T is a globally estimated sphg trace operator of degree $m \in \mathbb{Z}$ and class 0, then it is bounded

$$\text{(A.9)} \quad T : H^{s,\mu}(\overline{\mathbb{R}}^n_+) \to H^{s-m-\frac{1}{2},\mu}(\mathbb{R}^{n-1}), \quad \textit{uniformly in } \mu, \textit{ for } s \in \overline{\mathbb{R}}_+.$$

Proof. Consider an sphg Poisson operator K of degree m, written as in (3.21) with symbol-kernel $\tilde{k}(x', x_n, \xi', \mu)$. It satisfies (cf. Definition 3.3)

$$\sup_x |D^\beta_{x'}(1 + \langle \xi', \mu \rangle x_n) \tilde{k}(x', x_n, \xi', \mu)| = O(\langle \xi', \mu \rangle^{m+1}), \text{ all } \beta,$$

and hence

$$\|\mathrm{OP}'\big(\langle \xi', \mu \rangle^{-m-1}(1 + \langle \xi', \mu \rangle x_n) \tilde{k}(x', x_n, \xi', \mu)\big)\|_{\mathscr{L}(L_2(\mathbb{R}^{n-1}))} = O(1),$$

uniformly in $x_n \in \overline{\mathbb{R}}_+$ and μ. It follows that

$$\begin{aligned}
\|Kv\|^2_{L_2(\mathbb{R}^n_+)} &= \int_{\mathbb{R}^n_+} |\mathrm{OP}'\big(\tilde{k}(x', x_n, \xi', \mu)\big) v|^2 \, dx' dx_n \\
&= \int_{\mathbb{R}^{n-1}} \int_0^\infty |\mathrm{OP}'\big(\tilde{k}(x', x_n, \xi', \mu) \langle \xi', \mu \rangle^{-m-1}(1 + \langle \xi', \mu \rangle x_n)\big) \\
&\qquad \cdot \mathrm{OP}'\big(\langle \xi', \mu \rangle^{m+1}(1 + \langle \xi', \mu \rangle x_n)^{-1}\big) v|^2 \, dx_n dx' \\
&\leqq C \int_{\mathbb{R}^{n-1}} \int_0^\infty |\mathrm{OP}'\big(\langle \xi', \mu \rangle^{m+1}(1 + \langle \xi', \mu \rangle x_n)^{-1}\big) v|^2 \, dx_n dx' \\
&= C \int_{\mathbb{R}^{n-1}} \int_0^\infty |\langle \xi', \mu \rangle^{m+1}(1 + \langle \xi', \mu \rangle x_n)^{-1} \hat{v}(\xi')|^2 \, dx_n \, đ\xi' \\
&= C' \|\langle D', \mu \rangle^{m+\frac{1}{2}} v\|^2_{L_2(\mathbb{R}^{n-1})}.
\end{aligned}$$

In other words, K is uniformly bounded from $H^{m+\frac{1}{2},\mu}(\mathbb{R}^{n-1})$ to $L_2(\mathbb{R}^n_+)$.

One finds similarly (or by duality) that when T is an sphg trace operator of degree m and class 0, written in the form (3.22), then it is uniformly bounded as an operator from $L_2(\mathbb{R}^n_+)$ to $H^{-m-\frac{1}{2},\mu}(\mathbb{R}^{n-1})$.

This shows (A.8) and (A.9) for $s = 0$. To include general s, we can use the uniform homeomorphisms

$$\text{(A.10)} \quad (\langle D',\mu\rangle - iD_n)^t_+ : H^{s,\mu}(\overline{\mathbb{R}}^n_+) \xrightarrow{\sim} H^{s-t,\mu}(\overline{\mathbb{R}}^n_+), \quad t \in \mathbb{Z}, s \in \mathbb{R},$$

cf. [GK]; other variants of order-reducing operators from earlier works could likewise be used. Those in (A.10) are not classical ψdo's in n variables, but compose nicely with the above Poisson and trace operators anyway; in fact, $(\langle D',\mu\rangle - iD_n)^t_+K$ is an sphg Poisson operator of degree $m+t$ for any $t \in \mathbb{Z}$, and $T(\langle D',\mu\rangle - iD_n)^t_+$ is an sphg trace operator of order $m+t$ and class 0 when $t \leqq 0$. Application of the preceding result gives:

$$(\langle D',\mu\rangle - iD_n)^t_+K : H^{m+t+\frac{1}{2},\mu}(\mathbb{R}^{n-1}) \to L_2(\mathbb{R}^n_+), \text{ for } t \in \mathbb{Z};$$
$$T(\langle D',\mu\rangle - iD_n)^t_+ : L_2(\mathbb{R}^n_+) \to H^{m+t+\frac{1}{2},\mu}(\mathbb{R}^{n-1}), \text{ for } t \leqq 0.$$

By use of (A.10) we get from this the statements in the lemma for integer values of s, and they extend to $s \in \mathbb{R}$ resp. $s \in \overline{\mathbb{R}}_+$ by interpolation. □

Such mapping properties hold in much more general situations (in [G1], [GK], regularity $\geqq 0$ suffices), and (A.9) extends to $s > -\frac{1}{2}$. There is an order convention linked with these mapping properties, slightly different from the degree convention, cf. [BM], [G1].

For operators on the boundary we observe:

Lemma A.4. *Let* $S \in OP'(S^{-k,0}(\mathbb{R}^{n-1},\mathbb{R}^{n-1},\mathbb{R}_+) \cap S^{0,-k}(\mathbb{R}^{n-1},\mathbb{R}^{n-1},\mathbb{R}_+))$ *for some integer* $k \geqq 0$, *with global estimates in* x'. *Then for all* $s \in \mathbb{R}$, S *is bounded,*

$$\text{(A.11)} \qquad S : H^{s,\mu}(\mathbb{R}^{n-1}) \to H^{s+k,\mu}(\mathbb{R}^{n-1}), \textit{ uniformly in } \mu.$$

Proof. For $M \in \mathbb{N}$, $2M \geqq k$, we have that

$$\text{(A.12)} \quad S\langle D',\mu\rangle^{k-2M} \text{ and } \langle D',\mu\rangle^{k-2M}S \in \mathrm{OP}'(S^{-2M,0} \cap S^{0,-2M}),$$

by Lemmas 1.6 and 1.13. Composition of (A.12) to the right or left with $(1-\Delta')^M$ (where $\Delta' = \partial_1^2 + \ldots + \partial_{n-1}^2$) gives operators in $\mathrm{OP}'(S^{0,0})$, in view of Definition 1.1 and the standard composition rules for ψdo's. Multiplication of (A.12) by μ^{2M} gives operators in $\mathrm{OP}'(S^{0,0})$, likewise in view of Definition 1.1. Thus

$$\text{(A.13)} \qquad ((1-\Delta')^M + \mu^{2M})S\langle D',\mu\rangle^{k-2M} \text{ and}$$
$$\langle D',\mu\rangle^{k-2M}S((1-\Delta')^M + \mu^{2M}) \in \mathrm{OP}'(S^{0,0}).$$

It follows from the standard mapping property of ψdo's that the operators in (A.13) are continuous in $L_2(\mathbb{R}^{n-1})$, uniformly in μ. Then since

$$\begin{aligned} \langle D',\mu\rangle^{k-2M} &: H^{s,\mu}(\mathbb{R}^{n-1}) \xrightarrow{\sim} H^{s-k+2M,\mu}(\mathbb{R}^{n-1}),\\ (1-\Delta')^M + \mu^{2M} &: H^{s,\mu}(\mathbb{R}^{n-1}) \xrightarrow{\sim} H^{s-2M,\mu}(\mathbb{R}^{n-1}), \end{aligned}$$

for any $s \in \mathbb{R}$, uniformly in μ, one concludes by suitable choices of s that

$$S : H^{2M-k,\mu}(\mathbb{R}^{n-1}) \to H^{2M,\mu}(\mathbb{R}^{n-1}) \text{ and}$$

$$S : H^{-2M,\mu}(\mathbb{R}^{n-1}) \to H^{k-2M,\mu}(\mathbb{R}^{n-1}),$$

uniformly in μ. Since M can be taken arbitrarily large, it follows by interpolation that (A.11) holds for all $s \in \mathbb{R}$. □

The lemma applies to $\mathcal{S}_{\mathcal{B}}$ and its derivatives. They are of regularity 0 in the terminology of [G1], [GK]. The rules in Lemma A.3 and A.4 extend to operators on the sections of vector bundles over compact manifolds with boundary, by use of local trivializations.

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