Almost orthogonality and L^2-boundedness

A generalization of the T^*T-argument can be used to prove L^2-boundedness of large classes of integral operators.

Here we use it to prove the Calderón–Vaillancourt theorem, that pseudodifferential operators associated to symbols in

$$S^m_{\rho,\delta} := \{ a \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n) : \underbrace{|\partial_x^\beta \partial_\xi^\alpha a| \le C_{\alpha\beta} \langle \xi \rangle^{m-\rho|\alpha|+\delta|\beta|}}_{(3)} \}$$

are bounded on $L^2(\mathbb{R}^n)$ for $0 \le \delta \le \rho < 1$ and $m=0$.

As $S^0_{\rho,\delta} \subseteq S^0_{\rho,\rho}$ if $\delta \le \rho$, it is sufficient to show this for $S^0_{\rho,\rho}$, $0 \le \rho < 1$.

Key ingredient in the generalized T^*T-method is the Cotlar–Stein lemma.

Thm 1: Suppose $\{T_j\}_{j=1}^N \subseteq \mathcal{L}(L^2(\mathbb{R}^n))$ and $\{\gamma(j)\}_{j=-\infty}^\infty \subseteq [0,\infty)$ s.t. $\sum_j \gamma(j) =: A < \infty$ (21)

and
$$\|T_i^* T_j\| \le [\gamma(i-j)]^2 \qquad (22)$$
$$\|T_i T_j^*\| \le [\gamma(i-j)]^2$$

Then: $T = \sum_{j=1}^N T_j \in \mathcal{L}(L^2(\mathbb{R}^n))$, $\|T\| \le A$.

The idea is to decompose the symbol $a = \sum_j a_j$ into summands of disjoint support (if $|j-j'|$ sufficiently large), and to apply the Cotlar–Stein lemma to $T_j := op(a_j)$.

Operators with symbol in $S^m_{\rho,\rho}$ naturally occur as parametrices of non-elliptic operators

Cotlar-Stein lemma

THEOREM 1 (~~Cotlar's lemma~~). *Under the assumptions* (21) *and* (22) *above, the operator*

$$T = \sum_j T_j$$

satisfies

$$\|T\| \le A. \tag{23}$$

We have stated the theorem in its *a priori* form, where only finitely many T_j are involved. What is important is that the conclusion (23) gives a bound independent of the number of these T_j.

2.2 Proof. We begin with some preliminary remarks about norms of operators on L^2. First, $\|T^*\| = \|T\|$; to see this note that

$$\|T\| = \sup |\langle Tf, g\rangle|,$$

where the supremum is taken over all $f, g \in L^2$ of norm 1. Since $\langle Tf, g\rangle = \langle f, T^*g\rangle$, the assertion follows. Second, $\|T\|^2 = \|T^*T\|$, because on the one hand

$$\|T^*T\| \le \|T^*\| \cdot \|T\| = \|T\|^2,$$

and on the other hand

$$\|T\|^2 = \sup_{\|f\|_{L^2}=1} \langle Tf, Tf\rangle = \sup \langle T^*Tf, f\rangle \le \|T^*T\|.$$

In particular, when T is self-adjoint, $\|T\|^2 = \|T^2\|$, and by induction $\|T\|^m = \|T^m\|$, at least when m is a power of 2.‡ Using this with T replaced by T^*T (which is visibly self-adjoint), we have

$$\|T\|^{2m} = \|(T^*T)^m\|. \tag{24}$$

In proving the theorem, we shall use (24) because it allows us to most efficiently exploit our hypotheses. Written out in full,

$$(T^*T)^m = \sum_{i_1,\ldots,i_{2m}} T^*_{i_1} T_{i_2} T^*_{i_3} \cdots T_{i_{2m}}. \tag{25}$$

We shall estimate this sum by majorizing the norms of the individual summands.

First, associating the factors in each summand as

$$(T^*_{i_1} T_{i_2})(T^*_{i_3} T_{i_4}) \cdots (T^*_{i_{2m-1}} T_{i_{2m}}),$$

‡ This assertion holds for arbitrary positive integers m, but requires a further argument.

and using the first inequality in (22), we get

$$\|T^*_{i_1} T_{i_2} \cdots T_{i_{2m}}\| \le \gamma^2(i_1 - i_2) \cdot \gamma^2(i_3 - i_4) \cdots \gamma^2(i_{2m-1} - i_{2m}). \tag{26}$$

Alternatively, we can associate the factors as

$$T^*_{i_1}(T_{i_2} T^*_{i_3}) \cdots (T_{i_{2m-2}} T^*_{i_{2m-1}}) T_{i_{2m}}.$$

Then since $\|T_{i_1}\| \le \gamma(0) \le A$, and similarly $\|T_{i_{2m}}\| \le A$, we get

$$\|T^*_{i_1} T_{i_2} \cdots T_{i_{2m}}\| \le A^2 \cdot \gamma^2(i_2 - i_3) \cdot \gamma^2(i_4 - i_5) \cdots \gamma^2(i_{2m-2} - i_{2m-1}). \tag{27}$$

We take the geometric mean of (26) and (27) and insert this in (25). The result is

$$\|(T^*T)^m\| \le \sum_{i_1,\ldots,i_{2m}} A \cdot \gamma(i_1 - i_2) \cdot \gamma(i_2 - i_3) \cdots \gamma(i_{2m-1} - i_{2m}).$$

In the above, we first sum in i_1 and use the fact that $\sum_{i_1} \gamma(i_1 - i_2) \le A$. Next, we sum in i_2, using $\sum_{i_3} \gamma(i_2 - i_3) \le A$. Continuing in this way for the indices $i_1, \ldots, i_{2m-1}$ gives

$$\|(T^*T)^m\| \le A^{2m} \sum_{i_{2m}} 1.$$

We assumed that we had only finitely many nonzero T_i's; say there are N of them. Then, by (24),

$$\|T\| \le A \cdot N^{1/2m}.$$

Finally, we let $m \to \infty$, proving the theorem.

2.3 Remarks.

1. In the theorem above, the summation was taken over the integers. However, we can replace $\mathbf{Z}$ here by $\mathbf{Z}^r$ (for any r) and the result still holds, without any change in the proof. Indeed, if we write $j = (j_1, j_2, \ldots j_r) \in \mathbf{Z}^r$, $T = \sum_{j\in\mathbf{Z}^r} T_j$, and require $\sum_{j\in\mathbf{Z}^r} \gamma(j) = A < \infty$, then we can leave the assumptions (22) and conclusion (23) unchanged.

One may also obtain a variant of the theorem where the decomposition of T is given by integrals rather than sums. For such a formulation, see §5.5.

2. A special case of the theorem—actually, its original version—arises when the T_j are self-adjoint and mutually commuting. In that case the T_j can be given a simultaneous spectral resolution and the result thereby reduces to the easy situation where the T_j are scalar multiples of the identity. See §5.4.

3. We now describe a cruder version of the theorem, which can sometimes be used as a substitute in applications and whose proof is much more direct. We assume the T_j satisfy:

$$
\begin{aligned}
&\|T_j\| \le A, \\
&\|T_i^* T_j\| \le \gamma(i)\cdot\gamma(j), \quad \text{if } i \neq j, \text{ and} \\
&T_i T_j^* = 0, \qquad\qquad \text{if } i \neq j.
\end{aligned} \tag{28}
$$

As before, we suppose that $\sum_j \gamma(j) = A < \infty$.

PROPOSITION. *Under these assumptions, we have that*

$$\|\sum_j T_j\| \le 2^{1/2}\cdot A.$$

To prove this, observe that, because of (28), the ranges of the T_j^* are mutually orthogonal. In fact

$$\langle T_j^* f, T_i^* g\rangle = \langle T_i T_j^* f, g\rangle = 0, \qquad \text{if } i \neq j.$$

Therefore $T_j^* = E_j T_j^*$, where the E_j are mutually orthogonal projections, and hence also $T_j = T_j E_j$. As a result

$$\sum_j \|T_j(f)\|^2 \le A^2 \sum_j \|E_j(f)\|^2 \le A^2\|f\|^2.$$

Also if $T = \sum_j T_j$, then

$$\|T(f)\|^2 = \sum_{i,j}\langle T_j f, T_i f\rangle = \sum_j \|T_j(f)\|^2 + \sum_{i\neq j}\langle T_i^* T_j f, f\rangle.$$

We now invoke (28) again to get $\|Tf\|^2 \le 2A^2\|f\|^2$, which is the desired conclusion.

2.4 The class $S_{0,0}$. Our first application of almost orthogonality is the following result:

PROPOSITION. *Suppose a is a symbol that belongs to the class $S^0_{0,0}$. Then the operator T_a, initially defined on $\mathcal{S}$, has a bounded extension to an operator from $L^2(\mathbf{R}^n)$ to itself.*

We observe that $T_a = S\mathcal{F}$, where $(\mathcal{F}f)(\xi) = \widehat{f}(\xi)$ is the Fourier transform, and

$$Sf(x) = \int_{\mathbf{R}^n} a(x,\xi)e^{2\pi i x\cdot\xi} f(\xi)\, d\xi. \tag{29}$$

By Plancherel's theorem, it suffices to establish the L^2-boundedness of the operator S. Notice that, in view of the assumptions on a (that is, (3) with $\rho = \delta = 0$), the roles of x and ξ in S are perfectly symmetric.

We now decompose the ξ-space, and, because of the aforementioned symmetry, we also decompose the x-space in the same way. Moreover, the differential inequalities satisfied by the symbol a lead us to make these portions into sets of essentially unit size.

To be precise, we choose a smooth, nonnegative function ϕ that is supported in the unit cube

$$Q_1 = \{x : |x_j| \le 1 \text{ for } j = 1, \dots, n\},$$

and for which

$$\sum_{i\in\mathbf{Z}^n} \phi(x - i) = 1. \tag{30}$$

To construct such a ϕ, simply fix any smooth, nonnegative ϕ_0 that equals 1 on the cube $Q_{1/2} = 1/2\cdot Q_1$, and is supported in Q_1. Noting that $\sum_{i\in\mathbf{Z}^n}\phi_0(x-i)$ converges and is bounded away from 0 for all $x \in \mathbf{R}^n$, we take

$$\phi(x) = \phi_0(x)\left[\sum_{i\in\mathbf{Z}^n}\phi_0(x-i)\right]^{-1}.$$

Next, let $\mathbf{i} = (i, i') \in \mathbf{Z}^{2n} = \mathbf{Z}^n \times \mathbf{Z}^n$ denote an element of $\mathbf{Z}^{2n}$, and similarly write $\mathbf{j} = (j, j')$ for another element of $\mathbf{Z}^{2n}$. We set

$$a_{\mathbf{i}}(x,\xi) = \phi(x - i)\, a(x,\xi)\, \phi(\xi - i'),$$

and write $S_{\mathbf{i}}$ for the operator (29) with $a(x,\xi)$ replaced by $a_{\mathbf{i}}(x,\xi)$. In view of (30) we have that

$$S = \sum_{\mathbf{j}\in\mathbf{Z}^{2n}} S_{\mathbf{j}} \tag{31}$$

where, as is easily verified, for each $f \in \mathcal{S}$, the sum $\sum_{\mathbf{j}\in\mathbf{Z}^{2n}} S_{\mathbf{j}}(f)$ converges in $\mathcal{S}$ to $S(f)$.

The main point is then to verify the almost-orthogonality estimates:

$$\|S_{\mathbf{i}}^* S_{\mathbf{j}}\| \le A(1 + |\mathbf{i} - \mathbf{j}|)^{-2N} \tag{32}$$

and

$$\|S_{\mathbf{i}} S_{\mathbf{j}}^*\| \le A(1 + |\mathbf{i} - \mathbf{j}|)^{-2N}. \tag{33}$$

Here $\|\cdot\|$ denotes the L^2 operator norm, N is sufficiently large, and the bound A is independent of $\mathbf{i}$ and $\mathbf{j}$.

2.4.1 To deal with (32) and (33), it is useful to recall the following simple estimate of the norm of an operator S in terms of the size of its kernel.

LEMMA. *Suppose S is given by*

(Schur's test)

$$(Sf)(x) = \int s(x,y)\, f(y)\, dy,$$

where the kernel s satisfies

$$\sup_x \int |s(x,y)|\, dy \leq 1,$$

and

$$\sup_y \int |s(x,y)|\, dx \leq 1.$$

Then

$$\|S\|_{L^2 \to L^2} \leq 1. \tag{34}$$

The L^2 operator norm $\|S\|$ equals

$$\sup |\langle Sf, g\rangle| = \sup \left| \int\int s(x,y)\, f(y)\, \bar{g}(x)\, dy\, dx \right|,$$

where the supremum is taken over all f and g with $\|f\|_{L^2} \leq 1$ and $\|g\|_{L^2} \leq 1$. Since

$$|fg| \leq (|f|^2 + |g|^2)/2,$$

the integral in question is dominated by

$$\frac{1}{2}\left\{ \int\int |s(x,y)|\, |f(y)|^2\, dy\, dx + \int\int |s(x,y)|\, |g(x)|^2\, dy\, dx \right\}.$$

In the first integral, we carry out the integration first with respect to x; in the second integral, we integrate first with respect to y. Invoking our hypotheses then gives us the desired conclusion (24).

2.4.2 Observe that if

$$(S_{\mathbf{i}}^* S_{\mathbf{j}})(f)(\xi) = \int s_{\mathbf{ij}}(\xi,\eta)\, f(\eta)\, d\eta,$$

then the kernel of $S_{\mathbf{i}}^* S_{\mathbf{j}}$ is given by

$$s_{\mathbf{ij}}(\xi,\eta) = \int \bar{a}_{\mathbf{i}}(x,\xi)\, a_{\mathbf{j}}(x,\eta) e^{2\pi i x\cdot(\eta-\xi)}\, dx.$$

In the above integral we integrate by parts, using the identity

$$(I - \Delta_x)^N e^{2\pi i x\cdot(\eta-\xi)} = (1 + 4\pi^2|\eta-\xi|^2)^N e^{2\pi i x\cdot(\eta-\xi)}.$$

We also note that $a_{\mathbf{i}}(x,\xi)$ and $a_{\mathbf{j}}(x,\eta)$ are given by

$$\phi(x - i)\, a(x,\xi)\, \phi(\xi - i') \qquad \text{and} \qquad \phi(x-j)\, a(x,\eta)\, \phi(\eta - j')$$

respectively, and so have disjoint x-support, unless $i - j \in Q_1$. These observations lead to the bounds

$$\begin{cases} |s_{\mathbf{ij}}(\xi,\eta)| \leq \dfrac{A_N\, \phi(\xi - i')\, \phi(\eta - j')}{(1 + |\xi - \eta|)^{2N}} & \text{if } i - j \in Q_1, \\ s_{\mathbf{ij}}(\xi,\eta) = 0 \quad \text{if } i - j \notin Q_1. \end{cases}$$

So if we invoke the lemma, we get immediately that

$$\|S_{\mathbf{i}}^* S_{\mathbf{j}}\| \leq A(1 + |\mathbf{i} - \mathbf{j}|)^{-2N}.$$

Since, as we have noted, the situation is symmetric in x and ξ, the same proof also shows that

$$\|S_{\mathbf{i}} S_{\mathbf{j}}^*\| \leq A(1 + |\mathbf{i} - \mathbf{j}|)^{-2N},$$

which gives (32) and (33).

Now it is only a matter of applying the almost-orthogonality theorem in the version described in the first remark in §2.3. Here $r = 2n$ and $\gamma(\mathbf{j})$ is a multiple of $(1 + |\mathbf{j}|)^{-N}$, with $N > 2n$. The result is that

$$\left\|\sum S_{\mathbf{j}}\right\| \leq A,$$

where the sum is taken over any finite subset of $\mathbf{Z}^{2n}$, and the bound A is independent of that subset. Because of the convergence of $\sum S_{\mathbf{j}}(f)$ in $\mathcal{S}$ (whenever $f \in \mathcal{S}$) already alluded to, we may conclude that

$$\|S(f)\|_{L^2} \leq A\|f\|_{L^2}, \qquad f \in \mathcal{S}.$$

From this, the extendability of S, and therefore of T_a, to a bounded operator on L^2 is evident.

Remark. Note that the above argument gives a bound for T that depends only on the L^∞ norms of the derivatives (of order at most $4n+1$) of the symbol a. Incidentally, more complicated arguments allow one to reduce this order substantially (see §5.13).

2.5 The class $S^0_{\rho,\rho}$. We shall now obtain a similar result for the more general classes $S^0_{\rho,\rho}$, when $0 \le \rho < 1$.

THEOREM 2. *Suppose a is a symbol that belongs to $S^0_{\rho,\rho}$ with $0 \le \rho < 1$. Then the operator T_a (initially defined on $\mathcal{S}$), has a bounded extension from $L^2(\mathbf{R}^n)$ to itself.*

Proof. We begin by replacing the symbol $a(x,\xi)$ with

$$a_\varepsilon(x,\xi) = a(x,\xi)\,\gamma(\varepsilon x, \varepsilon y), \qquad 0 < \varepsilon \le 1;$$

here γ is a fixed smooth function of compact support, with $\gamma(0,0) = 1$. Notice that the symbols a_ε belong to $S^0_{\rho,\rho}$ uniformly in ε and that, for $f \in \mathcal{S}$,

$$T_{a_\varepsilon}(f) \to T_a(f), \quad \text{as } \varepsilon \to 0,$$

in the topology of $\mathcal{S}$. This will allow us to assume that our initial symbol has compact support in both x and ξ, so the manipulations appearing below will be automatically justified. Also, in what follows, the explicit dependence on ε will be suppressed, but all of our estimates will be made independent of ε.

We treat the operator T_a by using the standard dyadic partition of the ξ-space, which occurred already in §4.1 of the previous chapter. That is, we write

$$T_a = \sum_{j=0}^{\infty} T_{a_j} = T_a S_0 + \sum_{j=1}^{\infty} T_a \Delta_j, \tag{35}$$

where $a_0(x,\xi) = a(x,\xi)\,\eta(\xi)$ is supported in $|\xi| \le 2$, and, for $j \ge 1$, $a_j(x,\xi) = a(x,\xi)\,\widehat{\Psi}(2^{-j}\xi)$ is supported in the shell $2^{j-1} \le |\xi| \le 2^{j+1}$. Note that the symbols a_j are uniformly in $S^0_{\rho,\rho}$.

There are two distinct parts to the proof: first, that the operators T_{a_j} are essentially mutually orthogonal, and second, that they are uniformly bounded in norm. To proceed, it will be convenient to break the sum (35) into two parts

$$T_a = \sum_{j \text{ even}} T_{a_j} + \sum_{j \text{ odd}} T_{a_j},$$

so that the summands in each part have disjoint ξ-support; it suffices to prove the boundedness of each sum separately.

Let us first consider the sum taken over the odd j. We have that

$$T_{a_j} T^*_{a_k} = 0 \qquad \text{if } j \ne k \tag{36}$$

because $T_{a_j}(T_{a_k})^* = T_a\Delta_j(T_a\Delta_k)^* = T_a\Delta_j\Delta_k^* T_a$, and the supports of the multipliers corresponding to Δ_j and Δ_k are disjoint. Next, we estimate $T^*_{a_j}T_{a_k}$. Since

$$T_{a_j}(f)(z) = \int_{\mathbf{R}^n\times\mathbf{R}^n} a_j(z,\xi)e^{2\pi i\xi\cdot(z-y)} f(y)\,d\xi\,dy,$$

and

$$T^*_{a_k}(g)(x) = \int_{\mathbf{R}^n\times\mathbf{R}^n} \bar{a}_k(z,\eta)e^{2\pi i\eta\cdot(x-z)} g(z)\,dz\,d\eta,$$

(see §1.4 of the previous chapter), we see that

$$(T^*_{a_j}T_{a_k})f(x) = \int K(x,y)\,f(y)\,dy,$$

with

$$K(x,y) = \int_{\mathbf{R}^n\times\mathbf{R}^n\times\mathbf{R}^n} \bar{a}_k(z,\eta)\,a_j(z,\xi)e^{2\pi i[\xi\cdot(z-y)-\eta\cdot(z-x)]}\,dz\,d\eta\,d\xi. \tag{37}$$

To bound the kernel K, we exploit the oscillatory nature of the exponential (and the relative smoothness of the factors $\bar{a}_k$ and a_j) by integrating by parts with respect to the variables z, η, and ξ. First, one carries this out with respect to the z-variable by writing

$$\frac{(I-\Delta_z)^N}{(1+4\pi^2|\xi-\eta|^2)^N} e^{2\pi i(\xi-\eta)\cdot z} = e^{2\pi i(\xi-\eta)\cdot z},$$

inserting this in (37), and passing the z-differentiations to the factors a_k and a_j. Next, one performs a similar process on the η-variable, beginning with

$$\frac{(I-\Delta_\eta)^N}{(1+4\pi^2|x-z|^2)^N} e^{2\pi i\eta(x-z)} = e^{2\pi i\eta(x-z)},$$

and then passing the differentiations in the η-variable. Finally, an analogous step is carried out for the ξ-variable. If we take into account the differential inequalities for the symbols a_j (see (3)), and the restrictions on their supports, we see that each order of differentiation in the z-variable gives us a factor of order

$$(1+|\xi-\eta|)^{-1} \approx 2^{-\max(k,j)}$$

for every factor of order

$$(1+|\xi|+|\eta|)^{\rho} \approx 2^{\rho\max(k,j)}$$

that we may lose. As a result, the kernel K is dominated by a constant multiple of

$$2^{\max(k,j)\cdot 2\rho N} \cdot 2^{-\max(k,j)\cdot 2N} \cdot 2^{\max(k,j)\cdot 2n} \int_{\mathbf{R}^n} Q(x-z)\,Q(z-y)\,dz,$$

where $Q(z) = (1+|z|)^{-2N}$, if $k \ne j$.

Now if $K_0(x,y) = \int Q(x-z)\,Q(z-y)\,dz$, then

$$\int K_0(x,y)\,dy = \int K_0(x,y)\,dx = \left(\int (1+|z|)^{-2N}\right)^2 < \infty,$$

if $2N > n$. Thus, invoking the lemma in §2.4.1, we get

$$\|T^*_{a_j} T_{a_k}\| \le A \cdot 2^{\max(k,j)[2\rho N - 2N + 2n]}, \qquad j \ne k,$$

which implies that

$$\|T^*_{a_j} T_{a_k}\| \le \gamma(j)\,\gamma(k), \qquad j \ne k, \tag{39}$$

with $\gamma(j) = A \cdot 2^{-\varepsilon j}$, $\varepsilon > 0$, if we choose N so large that $N > n/(1-\rho)$; then $\varepsilon = N(1-\rho) - n$.

We have therefore satisfied the hypothesis (28) of the proposition in §2.3, save for what is here the most crucial step: that the summands in (35), namely the operators T_{a_j}, are uniformly bounded in norm.

To prove this, note what happens if we re-scale the symbols

$$a_j(x,\xi) = a(x,\xi)\,\Psi(2^{-j}\xi),$$

by setting

$$\tilde{a}_j(x,\xi) = a_j(2^{-j\rho}x, 2^{j\rho}\xi).$$

In view of the $S_{\rho,\rho}$ inequalities satisfied by the a_j, we observe that the $\tilde{a}_j$ belong to $S^0_{0,0}$, uniformly in j.

Next, if Λ_j denote the scaling operators given by

$$\Lambda_j(f)(x) = f(2^{j\rho}x),$$

then, as is easily verified,

$$T_{a_j} = \Lambda_j T_{\tilde{a}_j} \Lambda_j^{-1}.$$

Now $\|\Lambda_j f\|_{L^2} = 2^{nj\rho/2}\|f\|_{L^2}$ and $\|\Lambda_j^{-1} f\|_{L^2} = 2^{-nj\rho/2}\|f\|_{L^2}$; so the boundedness result for operators of the class $S^0_{0,0}$ (given in §2.4), applied to the $\tilde{a}_j$, guarantees that

$$\|T_{a_j}\| \le A. \tag{47}$$

We may therefore conclude that the sum $\sum_{j \text{ odd}} T_{a_j}$ yields a bounded operator on L^2; the sum $\sum_{j \text{ even}} T_{a_j}$ is treated similarly, and the theorem is therefore proved.

3. L^2 theory of operators with Calderón-Zygmund kernels

We now turn to the important problem of determining necessary and sufficient conditions for an operator, whose kernel satisfies differential inequalities of the Calderón-Zygmund type (i.e., of the nature (1)), to be extendable to a bounded operator on $L^2(\mathbf{R}^n)$. The various issues involved in this problem can be formulated in terms of two related questions.

We consider first an operator T, initially defined as a mapping from test functions in $\mathcal{S}$ to distributions in $\mathcal{S}'$, to which is associated a kernel $K(x,y)$, defined when $x \ne y$, that satisfies inequalities of the kind:

$$|\partial_x^\alpha \partial_y^\beta K(x,y)| \le A|x-y|^{-n-|\alpha|-|\beta|}; \tag{42}$$

such kernels are called *Calderón-Zygmund kernels* or *singular integral kernels.*†

The assumed relation between T and K is that, whenever $f \in \mathcal{S}$ has compact support, then the distribution Tf can be identified with the function

$$(Tf)(x) = \int K(x,y)\,f(y)\,dy, \tag{43}$$

for x outside the support of f.‡

QUESTION 1. *What additional conditions must be imposed on T to guarantee that it extends to a bounded operator from $L^2(\mathbf{R}^n)$ to itself?*

In the second version of our problem, our starting point is not a (densely defined) operator T, but rather a kernel $K(x,y)$, given for $x \ne y$.

QUESTION 2. *Suppose the function $K(x,y)$ satisfies the differential inequalities (42). What additional ("cancellation") conditions must be imposed on K so that there exists a bounded operator $T : L^2(\mathbf{R}^n) \to L^2(\mathbf{R}^n)$ having kernel K in the sense of (43)?*

3.1 Three simple propositions. The following observations will help clarify the nature of the conditions that will be imposed, as well as motivate some aspects of the proof of the main theorem. The first proposition shows graphically that a cancellation condition must be required for the kernel K.

PROPOSITION 1. *Suppose $K(x,y)$ is a function given for $x \ne y$ that satisfies the inequality $K(x,y) \ge c|x-y|^{-n}$, $c > 0$. Then there does not exist an operator T that is bounded on $L^2(\mathbf{R}^n)$ for which K is the kernel, in the sense of (43).*

† This terminology also applies to kernels satisfying a weaker version of these inequalities (such as (48) below).

‡ Note that the operator T is *not* uniquely determined by the kernel $K(x,y)$ (via (43)). A simple example is the identity operator (or, more generally, multiplication by a bounded function), for which the kernel is identically zero.