B. Simon, Orthogonal Polynomials … he unit circle, Part 2

Remarks and Historical Notes. There is a huge literature on decay of Green's function and eigenfunctions for ODEs and PDEs (see especially Agmon [**11**]). The approach we use here has its roots in part in the first proof of exponential decay for N-body Schrödinger operators by O'Connor [**829**]. Combes-Thomas [**211**] realized the right language for formulating O'Connor's result was that of operators analytic under a group action, a notion Combes and collaborators [**12, 76**] had used to study what has come to be called complex scaling. Propositions 10.14.4, 10.14.5, and 10.14.6 are abstractions of the Combes-Thomas approach which has been widely used. Their use in the context of OPUC is new (not surprising, given that it relies on the CMV matrix, which is of recent vintage), but should be regarded as a straightforward import.

The use of analytic vectors in the study of groups is due to Nelson [**804**].

In many cases, we expect the quantity $\tilde{\Theta}$ in (10.14.3) to go to zero as $(\text{dist}(z_0, (\text{supp}(d\mu\backslash\{z_0\}))^{1/2}$ but, in general, it seems unlikely that one can improve on the linear bound.

10.15. Counting Eigenvalues in Gaps: The Birman-Schwinger Principle

In this section, we want to discuss a situation that compares two sets of Verblunsky coefficients, $\{\alpha_n^{(0)}\}_{n=0}^\infty$ and $\{\alpha_n\}_{n=0}^\infty$, where $|\alpha_n - \alpha_n^{(0)}| \to 0$ with some information on the rate. Suppose that $d\mu^{(0)}$ and $d\mu$ are the associated probability measures on $\partial\mathbb{D}$ and that some open interval, I, is disjoint from $\sigma_{\text{ess}}(d\mu^{(0)})$, and so from $\sigma_{\text{ess}}(d\mu)$ by Theorem 4.3.8. Suppose also that $d\mu^{(0)}$ has only finitely many pure points in I. When is the same true for $d\mu$? If the number of pure points is infinite, the pure points can only have ∂I as limit points, and one can ask about the growth of the number of pure points in $\{z \mid \text{dist}(z, \partial\mathbb{D}\backslash I) > \varepsilon\}$ as $\varepsilon \downarrow 0$. In Section 12.2, we will apply this to the case where $\alpha_n^{(0)}$ is periodic, but here we will describe a general framework.

The analogous problem for Schrödinger operators has been heavily studied (see the Notes) and the techniques we use are borrowed from there. Indeed, it is not obvious how to carry the techniques over directly to unitaries, so we will proceed by using Cayley transforms to reduce to a problem in perturbations of selfadjoint operators. We thus begin by discussing this situation. The following preliminary is useful:

Proposition 10.15.1. *Let A and B be two bounded operators on a Banach space, X. Then*

$$\sigma(AB)\backslash\{0\} = \sigma(BA)\backslash\{0\} \tag{10.15.1}$$

Proof. We claim that if $\lambda \in \rho(AB)\backslash\{0\}$, then

$$-\lambda^{-1} + \lambda^{-1}B(AB-\lambda)^{-1}A \tag{10.15.2}$$

is a two-sided inverse for $(BA-\lambda)$ since $(BA-\lambda)B = B(AB-\lambda)$, so

$$(BA-\lambda)(10.15.2) = -\lambda^{-1}(BA-\lambda) + \lambda^{-1}(BA) = \mathbf{1}$$

Thus $\rho(BA)\backslash\{0\} \supset \rho(AB)\backslash\{0\}$, so $\sigma(BA)\backslash\{0\} \subset \sigma(AB)\backslash\{0\}$. Interchanging A and B, we get (10.15.1). □

Let $A_0 \geq 0$ be a bounded selfadjoint operator and let

$$A = A_0 + B \tag{10.15.3}$$

with B compact and selfadjoint. Thus, $\sigma_{\text{ess}}(A) = \sigma_{\text{ess}}(A_0)$, so in any interval $(-\infty, E)$ with $E < 0$, A has only finite point spectrum with each eigenvalue of finite multiplicity. Define

$$N_E(A) = \dim P_{(-\infty,E)}(A) \tag{10.15.4}$$

the number of eigenvalues less than E counting multiplicity. We want to get bounds on $N_E(A)$, especially ones that give us information as $E \uparrow 0$.

Proposition 10.15.2. *Let $A_0 \geq 0$ and B be compact and selfadjoint. For $\lambda \in \mathbb{R}$, define*

$$A_\lambda = A_0 + \lambda B \tag{10.15.5}$$

Let $e_j(\lambda)$ be the j-th eigenvalue (counting multiplicity) of A_λ, counting from the bottom with the convention $e_j(\lambda) = \inf \sigma_{\text{ess}}(A_0) \equiv e_\infty$ if there are not j eigenvalues. Then, for $\lambda > 0$,

(i) *In the region $\{\lambda \mid e_j(\lambda) < e_\infty\}$, e_j is strictly monotone decreasing, and if $e_j(\lambda_0) < e_\infty$, $e_j(\lambda) < e_\infty$ for all $\lambda > \lambda_0$.*

(ii) *$e_0 < 0$ is an eigenvalue of A_λ if and only if λ^{-1} is an eigenvalue of*

$$(e_0 - A_0)^{-1}B = K_1(e_0) \tag{10.15.6}$$

(iii)

$$N_E(A) = \#\{\mu \mid \mu > 1, \mu \in \sigma(K_1(E))\} \tag{10.15.7}$$

where we count geometric multiplicities.

(iv) *Let $B = CUD$. Then*

$$N_E(A) \leq \text{Tr}((C^*(A_0-E)^{-1}C)^q)^{r/2q}\|U\|^r\,\text{Tr}((D^*(A_0-E)^{-1}D)^p)^{r/2p} \tag{10.15.8}$$

for any $p, q \geq 1$ and $r = [\frac{1}{2}(p^{-1}+q^{-1})]^{-1}$.

Remarks. 1. Since $\sigma((A_0-e_0)^{-1}B)\backslash\{0\} = \sigma((A_0-e_0)^{-1/2}B(A_0-e_0)^{-1/2})\backslash\{0\}$ and the second operator is selfadjoint, $(A_0-e_0)^{-1}B$ only has real eigenvalues by Proposition 10.15.1. An easy extension of the argument shows that $(A_0-e_0)^{-1}B$ has no Jordan anomalies, that is, the geometric and algebraic multiplicities of its eigenvalues are equal.

2. (10.15.8) with $U = 1$ and $C = D = |B|^{1/2}$ and with $p = q = r$, that is,

$$N_E(A) \leq \text{Tr}((|B|^{1/2}(A_0-E)^{-1}|B|^{1/2})^r) \tag{10.15.9}$$

is called the Birman-Schwinger bound.

Proof. (i) If $A_\lambda\varphi_\lambda^{(j)} = e_j(\lambda)\varphi_\lambda^{(j)}$ with $\|\varphi_\lambda^{(j)}\| = 1$, then eigenvalue perturbation theory (see [**615, 899**]) says

$$\begin{aligned}\frac{d}{d\lambda}e_j(\lambda) &= \langle\varphi_\lambda^{(j)}, B\varphi_\lambda^{(j)}\rangle \\ &= \lambda^{-1}[e_j(\lambda) - \langle\varphi_\lambda^{(j)}, A_0\varphi_\lambda^{(j)}\rangle] \\ &< 0\end{aligned} \tag{10.15.10}$$

since $e_\infty - A_0 \leq 0$.

(ii) $(A_0 + \lambda B)\varphi = e_0\varphi$ if and only if $(e_0 - A_0)^{-1}B\varphi = \lambda^{-1}\varphi$ since $e - A_0$ is an invertible operator.

(iii) The $e_j(\lambda)$ are continuous and strictly monotone, so the number of j with $e_j < E$ is the number of λ with $0 < \lambda < 1$ and $e_j(A_0 + \lambda B) = e_0$. By (ii), this is the number of eigenvalues of $K_1(E)$ in $(1, \infty)$.

(iv) By Proposition 10.15.1, $\sigma(K_1(e_0)) = \sigma(K_2(e_0))$ where

$$K_2(e_0) = D(e_0 - A_0)^{-1}CU$$

by (10.15.7), for any $r \geq 1$,

$$\begin{aligned} N_E(A) &= \#\{\mu \mid \mu > 1,\, \mu \in \sigma(K_2(e_0))\} \\ &\leq \Big(\sum_{\mu \in \sigma(K_2(e_0))} |\mu|^r \Big) \\ &\leq \operatorname{Tr}(|K_2(e_0)|^r) \qquad \text{(by (1.4.49))} \end{aligned}$$

k $r = [\frac{1}{2}(p^{-1} + q^{-1})]^{-1}$,

$$K_2(e_0) = [D(e_0 - A_0)^{-1/2}][(e_0 - A_0)^{-1/2}C]U$$

by Hölder's inequality for trace ideals,

$$\begin{aligned} \|K_2(e_0)\|_r^r &\leq \|D(e_0 - A_0)^{-1/2}\|_{2p}^r \,\|(e_0 - A_0)^{-1/2}C\|_{2q}^r \,\|U\|^r \\ &= \operatorname{Tr}([D(z_0 - A_0)^{-1}D^*]^p)^{r/2p}\operatorname{Tr}([C^*(z_0 - A_0)^{-1}C]^q)^{r/2q}\,\|U\|^r \end{aligned}$$ □

Here are two consequences of this result. For simplicity, we state them with $D = |B|^{1/2}$.

PROPOSITION 10.15.3. *Let $A_0 \geq 0$ and B be compact and selfadjoint. Suppose that for some orthonormal basis, $\{\varphi_n\}_{n=1}^\infty$, and some $p \geq 1$,*

$$\lim_{e_0 \uparrow 0} \langle \varphi_n, (|B|^{1/2}(A_0 - e_0)^{-1}|B|^{1/2})^p \varphi_n \rangle = b_n \tag{10.15.11}$$

exists and

$$N = \sum_{n=1}^\infty b_n < \infty \tag{10.15.12}$$

Then

$$\dim P_{(-\infty,0)}(A) \leq N \tag{10.15.13}$$

Suppose that for some $r > 0$ and $p \geq 1$,

$$\operatorname{Tr}([|B|^{1/2}(A_0 - e_0)^{-1}|B|^{1/2}]^p) \leq c|e_0|^{-r} \tag{10.15.14}$$

Then for any $k > r$,

$$\sum_{\substack{E_j \in \sigma(A) \\ E_j < 0}} |E|_j^k < \infty \tag{10.15.15}$$

PROOF. (i) We will only use the case $p = 1$, so we only give the details in that where monotonicity simplifies the argument.
Since $A_0 \geq 0$ if $e_0 < e_1 < 0$, then

$$0 \leq (A_0 - e_0)^{-1} \leq (A_0 - e_1)^{-1}$$

he left side of (10.15.11) is monotone increasing in e_0. Thus, for any M and < 0,

$$\sum_{n=1}^M \langle \varphi_n, (|B|^{1/2}(A_0 - e_0)^{-1}|B|^{1/2})\varphi_n \rangle \leq N$$

aking $M \to \infty$ and using (10.15.9), for any $E < 0$,

$$N_E(A) \leq N$$

which implies (10.15.13).

(ii) By (10.15.9) and (10.15.14), we have

$$N_E(A) \leq c|E|^{-r} \tag{10.15.16}$$

For any $e < 0$,

$$|e|^k = \int_e^0 k|y|^{k-1}\, dk$$

Thus

$$\sum_{e_j \leq e_0} |e_j|^k = \int_{e_1}^{e_2} k|y|^{k-1} + 2\int_{e_2}^{e_3} k|y|^{k-1}\, dy + \cdots$$

or

$$\begin{aligned} \sum_{e_j \leq e_0} |e_j|^k &\leq \int_{e_1}^{e_0} N_y(A) k |y|^{k-1}\, dy \\ &\leq ck \int_{e_1}^{e_0} |y|^{k-r-1}\, dy \\ &= ck(k-r)^{-1}[|e_1|^{k-r} - |e_0|^{k-r}] \\ &\leq ck(k-r)^{-1}|e_1|^{k-r} < \infty \end{aligned}$$

since $k > r$. Now take $e_0 \uparrow 0$. □

As an example, we have the following:

THEOREM 10.15.4. *Let J be a Jacobi matrix and suppose that $\sum n[|b_n| + (a_n - 1)_+] < \infty$. Then*

$$\dim P_{\mathbb{R}\setminus[-2,2]}(J) \leq \sum_{n=1}^\infty n|b_n| + (4n+2)(a_n - 1)_+ \tag{10.15.17}$$

and, in particular, it is finite.

Remark. By x_+, we mean $\max(x, 0)$ and $x_- = -\min(x, 0)$.

PROOF. We will prove that

$$\dim P_{(-\infty,-2)}(J) \leq \sum_{n=1}^\infty n(b_n)_- + (2n+1)(a_n - 1)_+ \tag{10.15.18}$$

This and a similar bound on $-J$ yields (10.15.17). Let $\tilde{J}$ be the Jacobi matrix with a_n replaced by $\min(a_n, 1)$ and b_n by $-[(b_n)_- + (a_n - 1)_+ + (a_{n-1} - 1)_+]$. We first note that since $(b_n)_- \leq b_n$ and

$$|2(a_n - 1)_+ u_n u_{n+1}| \leq (a_n - 1)_+(|u_n|^2 + |u_{n+1}|^2)$$

we have that

$$J \geq \tilde{J}$$

Thus

$$\dim P_{(-\infty,-2)}(J) \leq \dim P_{(-\infty,-2)}(\tilde{J}) \tag{10.15.19}$$

Since

$$\sum_{n=1}^\infty n[(b_n)_- + (a_n - 1)_+ + (a_{n-1} - 1)_+] = \text{RHS of (10.15.18)}$$

we need only prove the result for $\tilde{J}$, that is, for J's with $a_n \leq 1$ and $b_n \leq 0$.

Given such a J, let $\tilde{J}_0$ be the Jacobi matrix with the same values of a_n but = 0. Since $a_n \leq 1$, we claim that for any $e \leq -2$,

$$[(\tilde{J}_0 - e)^{-1}]_{nn} \leq [(J_0 - e)^{-1}]_{nn} \tag{10.15.20}$$

epting this, we finish the proof by noting that, by (1.2.24),

$$\lim_{e \uparrow -2} [(J_0 - e)^{-1}]_{nn} = \lim_{z \downarrow -1} -(z^{-1} - z)^{-1}[1 - z^{2n}] = n$$

by (10.15.20),

$$\lim_{e \uparrow -2} \langle \delta_n, |\tilde{J} - \tilde{J}_0|^{1/2}(\tilde{J}_0 - e)^{-1}|\tilde{J} - \tilde{J}_0|^{1/2}\delta_n\rangle \leq n(b_n)_-$$

By Proposition 10.15.3(i) and (10.15.19), we have proven (10.15.18) as required. To prove (10.15.20), we use a maximum principle argument. If $(U\varphi)_n =$ $)^n\varphi_n$, then $U\tilde{J}U^{-1} = -\tilde{J}$, so it suffices to prove for $e > 2$, then

$$[(e - \tilde{J}_0)^{-1}]_{nn} \leq [(e - J_0)^{-1}]_{nn} \tag{10.15.21}$$

:e $\|\tilde{J}_0\| \leq 2$, $\|J\| \leq 2$, $(e - \tilde{J}_0)^{-1} = \sum_{k=0}^{\infty} e^{-k-1}(\tilde{J}_0)^k$ shows that for all n, m, $-\tilde{J}_0)^{-1}]_{nm} \geq 0$. Since

$$\frac{\partial}{\partial a_k}[(e_0 - \tilde{J}_0)^{-1}]_{nn} = 2(e_0 - \tilde{J}_0)^{-1}_{nk}(e_0 - \tilde{J}_0)^{-1}_{k+1,n}$$

conclude $(e_0 - \tilde{J}_0)^{-1}_{nn}$ is monotone in each a_k, so (10.15.20) holds. □

Remark. We could use Proposition 10.15.3(i) to show that if $\sum|a_n - 1| + |b_n| <$ then $\sum_{n,\pm}(E_n^{\pm} - |2|)^p < \infty$ for all $p > \frac{1}{2}$, but we will not provide the details e we will prove a stronger result in Theorem 13.8.10.

Next, we want to discuss what happens when A_0 is no longer nonnegative but a finite number of eigenvalues in $(-\infty, 0)$. Since we will only use $p = q = r = 1$ pplications, we only state the result in that case, although there is a general ılt.

PROPOSITION 10.15.5. *Suppose A_0 is a selfadjoint operator so that $P_{(-\infty,0)}(A_0)$ nite rank and A_0 has m negative eigenvalues $E_1 \leq E_2 \leq \cdots \leq E_m \leq 0$. Let B ompact and selfadjoint, and $A = A_0 + B$. Suppose that*

$$B = CUD \tag{10.15.22}$$

$N_E(A)$ be $\dim P_{(-\infty,E)}(A)$ for $E_m < E < 0$. Then

$$N_E(A) \leq m + \|U\| c^{1/2} d^{1/2} \tag{10.15.23}$$

re

$$c = \mathrm{Tr}(C^*(A_0 - E)^{-1}C) + 2|E_m - E|^{-1}\mathrm{Tr}(C^*C) \tag{10.15.24}$$

$$d = \mathrm{Tr}(D(A_0 - E)^{-1}D^*) + 2|E_m - E|^{-1}\mathrm{Tr}(DD^*) \tag{10.15.25}$$

In particular, if

$$\limsup_{E \uparrow 0} |\mathrm{Tr}(D(A_0 - E)^{-1}D^*)| + |\mathrm{Tr}(C^*(A_0 - E)^{-1}C)| < \infty \tag{10.15.26}$$

ı $\dim P_{(-\infty,0)}(A)$ is finite, and if

$$|\mathrm{Tr}(D(A_0 - E)^{-1}D^*)| + |\mathrm{Tr}(C^*(A_0 - E)^{-1}C)| \leq C|E|^{-k}$$

for some $r > 0$, then (10.15.15) holds for any $k > r$.

PROOF. The eigenvalues $e_j(\lambda)$ of $A_0 + \lambda B$ are no longer monotone in λ; but it is still true that to increase the number of eigenvalues below E, there must be λ's with $0 < \lambda$ and $e_j(\lambda) = E$. Since eigenvalues can move in either direction across E, the number of such λ's may overcount, but it will always be an upper bound, that is,

$$N_E(A) \leq \#\{\mu > 1 \mid \mu \in \sigma((E_0 - A)^{-1}B)\}$$

Thus, following the proof of Proposition 10.15.2, we obtain (10.15.23) where

$$c = \|C^*(A_0 - e)^{-1}C\|_1 \tag{10.15.27}$$

and a similar formula for d (with $\|\cdot\|_1 = \mathrm{Tr}(|\cdot|)$).

Let $P_- = P_{(-\infty,0)}(A)$ and $P_+ = 1 - P_-$. Then

$$c \leq \|C^*(A_0 - E)^{-1}P_+C\|_1 + \|C^*(A_0 - E)^{-1}P_-C\|_1 = \mathrm{Tr}(C^*(A_0 - E)^{-1}C) - 2\mathrm{Tr}(C^*(A_0 - E)^{-1}P_-C)$$

so (10.15.23) follows from (10.15.27) and

$$-\mathrm{Tr}(C^*(A_0 - E)^{-1}P_-C) \leq \mathrm{Tr}(C|E - A_0|^{-1/2}P_-CC^*P_-|E - A_0|^{1/2}) \leq |E - E_m|^{-1}\mathrm{Tr}(C^*C) \tag{10.15.28}$$

where we used Proposition 10.15.1 to get (10.15.28).

The two "in particular" conclusions follow from (10.15.23) by repeating the arguments in the proof of Proposition 10.15.3. □

Now, we want to consider two unitaries U_0 and U so $U - U_0$ is compact. Eventually, they will be CMV matrices associated to two sets, $\{\alpha_n^{(0)}\}_{n=0}^{\infty}$ and $\{\alpha_n\}_{n=0}^{\infty}$, of Verblunsky coefficients. Consider a gap G in $\sigma_{\mathrm{ess}}(U) = \sigma_{\mathrm{ess}}(U_0)$ and suppose U_0 has only finitely many eigenvalues in G. By replacing U, U_0 by $e^{-i\theta_0}U_0, e^{-i\theta}U$, we can suppose 1 is in the gap and $1 \notin \sigma(U_0) \cup \sigma(U)$. Thus, $G = \{e^{i\theta} \mid -\theta_1 < \theta < \theta_2\}$, and we will focus on eigenvalues in $\tilde{G} = \{e^{i\theta} \mid 0 < \theta < \theta_2\}$. By replacing U_0, U by U_0^*, U^*, we can also analyze what happens on $\{e^{i\theta} \mid -\theta_1 < \theta < 0\}$.

Define Cayley transforms:

$$A_0 = \frac{i(1 + U_0)}{(1 - U_0)} + \cot\left(\frac{\theta_2}{2}\right) \tag{10.15.29}$$

$$A = \frac{i(1 + U)}{(1 - U)} + \cot\left(\frac{\theta_2}{2}\right) \tag{10.15.30}$$

The map

$$f(e^{i\theta}) \equiv \frac{i(1 + e^{i\theta})}{(1 - e^{i\theta})} + \cot\left(\frac{\theta_2}{2}\right) = -\cot\left(\frac{\theta}{2}\right) + \cot\left(\frac{\theta_2}{2}\right)$$

maps $\{e^{i\theta} \mid \theta_2 \leq \theta < 2\pi\}$ to $[0, \infty)$ and $\tilde{G}$ to $(-\infty, 0)$, so $P_{(-\infty,0)}(A_0) = P_{\tilde{G}}(U_0)$ is finite-dimensional and Proposition 10.15.5 applies to control the number of eigenvalues of U in $\{e^{i\theta} \mid 0 < \theta < \theta_2 - \varepsilon\}$ for each $\varepsilon > 0$, and perhaps as $\varepsilon \downarrow 0$.

We define $B = A - A_0$. Since

$$\frac{i(1 + z)}{1 - z} = -i + \frac{2i}{1 - z}$$