$\|T\psi_n\| \geq \lambda/2$. Since $\psi_n \xrightarrow{w} 0$, $T\psi_n \to 0$ by Theorem VI.11. Thus, $\lambda = 0$. As a result

$$\sum_{j=1}^{n} (\varphi_j, \cdot)T\varphi_j \to T$$

in norm since λ_n is just the norm of the difference. ∎

We have discussed a wide variety of properties of compact operators but we have not yet described any property which explains our special interest in them. The basic principle which makes compact operators important is the Fredholm alternative: If A is compact, then either $A\psi = \psi$ has a solution or $(I - A)^{-1}$ exists. This is not a property shared by all bounded linear transformations. For example, if A is the operator $(A\varphi)(x) = x\varphi(x)$ on $L^2[0, 2]$, then $A\varphi = \varphi$ has no solutions but $(I - A)^{-1}$ does not exist (as a bounded operator). In terms of "solving equations" the Fredholm alternative is especially nice: It tells us that if for any φ there is at most one ψ with $\psi = \varphi + A\psi$, then there is always exactly one. That is, compactness and uniqueness together imply existence; for an example, see the discussion of the Dirichlet problem at the end of the section.

As one might expect, since the Fredholm alternative holds for finite-dimensional matrices, it is possible to prove the Fredholm alternative for compact operators (in the Hilbert space case) by using the fact that any compact operator A can be written as $A = F + R$ where F has finite rank and R has small norm. Compactness combines very nicely with analyticity so we first prove an elegant result which is of great use in itself (see Sections XI.6, XI.7, XIII.4, and XIII.5).

Theorem VI.14 (analytic Fredholm theorem) Let D be an open connected subset of $\mathbb{C}$. Let $f: D \to \mathscr{L}(\mathscr{H})$ be an analytic operator-valued function such that $f(z)$ is compact for each $z \in D$. Then, either

(a) $(I - f(z))^{-1}$ exists for no $z \in D$.

or

(b) $(I - f(z))^{-1}$ exists for all $z \in D \backslash S$ where S is a discrete subset of D (i.e. a set which has no limit points in D). In this case, $(I - f(z))^{-1}$ is meromorphic in D, analytic in $D \backslash S$, the residues at the poles are finite rank operators, and if $z \in S$ then $f(z)\psi = \psi$ has a nonzero solution in $\mathscr{H}$.

Proof We will prove that near any z_0 either (a) or (b) holds. A simple connectedness argument allows one to convert this into a statement about all of D

(Problem 21). Given $z_0 \in D$, choose an r so that $|z - z_0| < r$ implies $\|f(z) - f(z_0)\| < \frac{1}{2}$ and pick F, an operator with finite rank so that

$$\|f(z_0) - F\| < \tfrac{1}{2}$$

Then, for $z \in D_r$, the disc of radius r about z_0, $\|f(z) - F\| < 1$. By expanding in a geometric series we see that $(I - f(z) + F)^{-1}$ exists and is analytic.

Since F has finite rank, there are independent vectors $\psi_1, \ldots, \psi_N$ so that $F(\varphi) = \sum_{i=1}^{N} \alpha_i(\varphi)\psi_i$. The $\alpha_i(\cdot)$ are bounded linear functionals on $\mathscr{H}$ so by the Riesz lemma there are vectors $\phi_1, \ldots, \phi_N$ so that $F(\varphi) = \sum_{n=1}^{N} (\phi_i, \varphi)\psi_i$ for all $\varphi \in \mathscr{H}$. Let $\phi_n(z) = ((I - f(z) + F)^{-1})^* \phi_n$ and

$$g(z) = F(I - f(z) + F)^{-1} = \sum_{n=1}^{N} (\phi_n(z), \cdot)\psi_n$$

By writing

$$(I - f(z)) = (I - g(z))(I - f(z) + F)$$

we see that $I - f(z)$ is invertible for $z \in D_r$ if and only if $I - g(z)$ is invertible and that $\psi = f(z)\psi$ has a nonzero solution if and only if $\varphi = g(z)\varphi$ has a solution.

If $g(z)\varphi = \varphi$, then $\varphi = \sum_{n=1}^{N} \beta_n \psi_n$ and the β_n satisfy

$$\beta_n = \sum_{m=1}^{N} (\phi_n(z), \psi_m)\beta_m \tag{VI.5a}$$

Conversely, if (VI.5a) has a solution $\langle \beta_1, \ldots, \beta_N \rangle$, then $\varphi = \sum_{n=1}^{N} \beta_n \psi_n$ is a solution of $g(z)\varphi = \varphi$. Thus $g(z)\varphi = \varphi$ has a solution if and only if the determinant

$$d(z) = \det\{\delta_{nm} - (\phi_n(z), \psi_m)\} = 0$$

Since $(\phi_n(z), \psi_m)$ is analytic in D_r so is $d(z)$ which means that either $S_r = \{z \mid z \in D_r, d(z) = 0\}$ is a discrete set in D_r or $S_r = D_r$. Now, suppose $d(z) \neq 0$. Then, given ψ, we can solve $(I - g(z))\varphi = \psi$ by setting $\varphi = \psi + \sum_{n=1}^{N} \beta_n \psi_n$ if we can find β_n satisfying

$$\beta_n = (\phi_n(z), \psi) + \sum_{m=1}^{N} (\phi_n(z), \psi_m)\beta_m \tag{VI.5b}$$

But, since $d(z) \neq 0$, this equation has a solution. Thus $(I - g(z))^{-1}$ exists if and only if $z \notin S_r$.

The meromorphic nature of $(I - f(z))^{-1}$ and the finite rank residues follow from the fact that there is an explicit formula for the β_n in (VI.5b) in terms of cofactor matrices. ▮

This theorem has four important consequences:

Corollary (the Fredholm alternative) If A is a compact operator on $\mathscr{H}$, then either $(I - A)^{-1}$ exists or $A\psi = \psi$ has a solution.

Proof Take $f(z) = zA$ and apply the last theorem at $z = 1$. ▮

Theorem VI.15 (Riesz–Schauder theorem) Let A be a compact operator on $\mathscr{H}$, then $\sigma(A)$ is a discrete set having no limit points except perhaps $\lambda = 0$. Further, any nonzero $\lambda \in \sigma(A)$ is an eigenvalue of finite multiplicity (i.e. the corresponding space of eigenvectors is finite dimensional).

Proof Let $f(z) = zA$. Then $f(z)$ is an analytic compact operator-valued function on the entire plane. Thus $\{z \mid zA\psi = \psi \text{ has a solution } \psi \neq 0\}$ is a discrete set (it is not the entire plane since it does not contain $z = 0$) and if $1/\lambda$ is not in this discrete set then

$$(\lambda - A)^{-1} = \frac{1}{\lambda}\left(I - \frac{1}{\lambda}A\right)^{-1}$$

exists. The fact that the nonzero eigenvalues have finite multiplicity follows immediately from the compactness of A. ▮

Theorem VI.16 (the Hilbert–Schmidt theorem) Let A be a self-adjoint compact operator on $\mathscr{H}$. Then, there is a complete orthonormal basis, $\{\phi_n\}$, for $\mathscr{H}$ so that $A\phi_n = \lambda_n \phi_n$ and $\lambda_n \to 0$ as $n \to \infty$.

Proof For each eigenvalue of A choose an orthonormal basis for the set of eigenvectors corresponding to the eigenvalue. The collection of all these vectors, $\{\phi_n\}$, is an orthonormal set since eigenvectors corresponding to distinct eigenvalues are orthogonal. Let $\mathscr{M}$ be the closure of the span of $\{\phi_n\}$. Since A is self-adjoint and $A\colon \mathscr{M} \to \mathscr{M}$, $A\colon \mathscr{M}^\perp \to \mathscr{M}^\perp$. Let $\tilde{A}$ be the restriction of A to $\mathscr{M}^\perp$. Then $\tilde{A}$ is self-adjoint and compact since A is. By the Riesz–Schauder theorem, if any $\lambda \neq 0$ is in $\sigma(\tilde{A})$, it is an eigenvalue of $\tilde{A}$ and thus of A. Therefore the spectral radius of $\tilde{A}$ is zero since the eigenvectors of A are in $\mathscr{M}$. Because $\tilde{A}$ is self-adjoint, it is the zero operator on $\mathscr{M}^\perp$ by Theorem VI.6. Thus, $\mathscr{M}^\perp = \{0\}$ since if $\varphi \in \mathscr{M}^\perp$, then $A\varphi = 0$ which implies that $\varphi \in \mathscr{M}$. Therefore, $\mathscr{M} = \mathscr{H}$.

The fact that $\lambda_n \to 0$ is a consequence of the first part of the Riesz–Schauder theorem which says that each nonzero eigenvalue has finite multiplicity and the only possible limit point of the λ_n is zero. ▮

Theorem VI.17 (canonical form for compact operators) Let A be a compact operator on $\mathscr{H}$. Then there exist (not necessarily complete) orthonormal sets $\{\psi_n\}_{n=1}^N$ and $\{\phi_n\}_{n=1}^N$ and positive real numbers $\{\lambda_n\}_{n=1}^N$ with $\lambda_n \to 0$ so that

$$A = \sum_{n=1}^{N} \lambda_n(\psi_n, \cdot)\phi_n \tag{VI.6}$$

The sum in (VI.6), which may be finite or infinite, converges in norm. The numbers, $\{\lambda_n\}$, are called the **singular values** of A.

Proof Since A is compact, so is A^*A (Theorem VI.12). Thus A^*A is compact and self-adjoint. By the Hilbert–Schmidt theorem, there is an orthonormal set $\{\psi_n\}_{n=1}^N$ so that $A^*A\psi_n = \mu_n\psi_n$ with $\mu_n \neq 0$ and so that A^*A is the zero operator on the subspace orthogonal to $\{\psi_n\}_{n=1}^N$. Since A^*A is positive, each $\mu_n > 0$. Let λ_n be the positive square root of μ_n and set $\phi_n = A\psi_n/\lambda_n$. A short calculation shows that the ϕ_n are orthonormal and that

$$A\psi = \sum_{n=1}^{N} \lambda_n(\psi_n, \psi)\phi_n \quad \blacksquare$$

The proof shows that the singular values of A are precisely the eigenvalues of $|A|$.

We conclude with a classical example.

Example (Dirichlet problem) The main impetus for the study of compact operators arose from the use of integral equations in attempting to solve the classical boundary value problems of mathematical physics. We briefly describe this method. Let D be an open bounded region in $\mathbb{R}^3$ with a smooth boundary surface ∂D. The Dirichlet problem for Laplace's equation is: given a continuous function f on ∂D, find a function u, twice differentiable in D and continuous on $\bar{D}$, which satisfies

$$\begin{aligned} \Delta u(x) &= 0 \qquad && x \in D \\ u(x) &= f(x) && x \in \partial D \end{aligned}$$

Let $K(x, y) = (x - y, n_y)/2\pi|x - y|^3$ where n_y is the outer normal to ∂D at the point $y \in \partial D$. Then, as a function of x, $K(x, y)$ satisfies $\Delta_x K(x, y) = 0$ in the interior which suggests that we try to write u as a superposition

$$u(x) = \int_{\partial D} K(x, y)\varphi(y)\, dS(y) \tag{VI.6a}$$

where $\varphi(y)$ is some continuous function on ∂D and dS is the usual surface measure. Indeed, for $x \in D$, the integral makes perfectly good sense and