INSTITUT FOR MATEMATISKE FAG
KØBENHAVNS UNIVERSITET
Heiko Gimperlein, Kim Petersen

November 3, 2010

PDL — In–class Examination

Problem 1 *(40 points)*

Consider the following initial and boundary value problem:

$$\begin{aligned} \partial_t u(t,x) - u(t,x) &= \partial_x^2 u(t,x) \quad (t>0,\ x \in (0, \tfrac{\pi}{2}))\ , \\ u(0,x) &= f(x) \quad (x \in [0, \tfrac{\pi}{2})]) \ , \\ u(t,0) = u(t, \tfrac{\pi}{2}) &= 0 \quad (t > 0)\ . \end{aligned}$$

a) Use the separation of variables method to find a Fourier series solution to this equation for the particular initial condition $f(x) = x$ ($x \in [0, \frac{\pi}{2}]$). In particular, you should show that the sine expansion of f is $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin(2nx)$.

Hints: You may use intermediate results from the exercises, lectures or the textbook, but do not expect to find a solution of this particular equation in these sources. You do NOT have to prove that your series solution is really differentiable with respect to t or (twice) x and satisfies the equation.

b) Does the sine expansion of the f from part a) converge pointwise or uniformly for $x \in [0, \frac{\pi}{2}]$? And what does it converge to?

c) Use Parseval's theorem and the sine expansion of f from part a) to show $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

d) Given a solution to part a), determine a solution to the following initial value problem with inhomogeneous boundary conditions

$$\begin{aligned} \partial_t u(t,x) - u(t,x) &= \partial_x^2 u(t,x) \quad (t>0,\ x \in (0, \tfrac{\pi}{2}))\ , \\ u(0,x) &= x + \cos(x) \quad (x \in [0, \tfrac{\pi}{2})]) \ , \\ u(t,0) = 1\ , \quad u(t, \tfrac{\pi}{2}) &= 0 \quad (t > 0)\ . \end{aligned}$$

Note that $u_s(t,x) = \cos(x)$ satisfies $\partial_t u_s(t,x) - u_s(t,x) = \partial_x^2 u_s(t,x)$ and the boundary conditions.

Problem 2 *(20 points)*

a) Let $u : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}$ be a solution of the equation

$$\Delta u(x,y) := \partial_x^2 u(x,y) + \partial_y^2 u(x,y) = \frac{u(x,y)}{x^2+y^2} \qquad ((x,y) \in \mathbb{R}^2 \setminus \{0\})$$

which only depends on the radial variable $r = \sqrt{x^2+y^2}$: $u(x,y) = \Phi(\sqrt{x^2+y^2})$.
Show that $\Phi(r) = ar + br^{-1}$ for some $a, b \in \mathbb{R}$.

Hint: If you express $\Delta u = \ldots$ in polar coordinates, you obtain an Euler differential equation.

b) Let $f \in \mathcal{S}(\mathbb{R})$. Show that

$$\mathcal{F}\left\{(x + \partial_x) f\right\}(\omega) = i(\omega + \partial_\omega)\mathcal{F}f(\omega)\ .$$

What is $\mathcal{F}\{(x+\partial_x)^{17} e^{-\frac{x^2}{2}}\} := \mathcal{F}\{\underbrace{(x+\partial_x)\cdots(x+\partial_x)}_{17 \text{ times}} e^{-\frac{x^2}{2}}\}$?

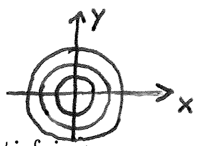
Problem 3 *(18 points)*

Give a SHORT answer to each of the following questions (≤ 3 lines per problem).
Please, avoid to spend much time on any of them.

a) Is the sum of periodic functions periodic? Give a proof or counterexample!

b) Is $\partial_x^2 u(x,y) + x\partial_y^2 u(x,y) = 0$ $((x,y) \in \mathbb{R}^2)$ a homogeneous, linear partial differential equation? Is it elliptic on $\mathbb{R}^2$?

c) The characteristic curves of the vector field $(-y, x)$ on $\mathbb{R}^2$ are circles centered around $(0,0)$:

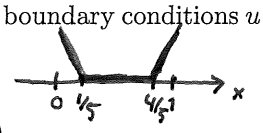


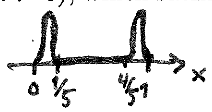
Why does the equation $-y\partial_x u(x,y) + x\partial_y u(x,y) = 0$ $((x,y) \in \mathbb{R}^2)$ not have a solution satisfying $u(x,0) = x$ for all $x \in \mathbb{R}$?

d) Let $f : \mathbb{R} \to \mathbb{R}$ continuously differentiable and 2π–periodic. Why is

$$\frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \int_{-\pi}^{\pi} f(x)\, e^{-inx}\, e^{iny}\, dx = f(y) \ ?$$

e) Let $u : [0,\infty) \times [0,1] \to \mathbb{R}$ be a solution to the wave equation $u_{tt}(t,x) = u_{xx}(t,x)$ $(t > 0, x \in (0,1))$ with homogeneous boundary conditions $u(t,0) = u(t,1) = 0$ $(t > 0)$, which satisfies the initial conditions:

$u(0,x) =$ 

, $u_t(0,x) =$ 

.

Determine $u(\frac{1}{10}, \frac{1}{2})$.

f) Let α_j be the j–th positive zero of the Bessel function J_0. Prove or disprove:

$$\int_{-1}^{1} \int_{0}^{1} \int_{0}^{2\pi} e^{-i\theta}\, J_0(\alpha_1 r)\, J_0(\alpha_3 r)\, e^{z^2 + z|\theta|}\, r\, d\theta\, dr\, dz = 1 \ .$$