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PDL — Exercises **8**

Fourier transforms and applications

Problem 39 *Calculating Fourier transforms*
Compute the Fourier transforms of the following functions:

a) $f(x) = 1 - x^2$ if $|x| < 1$ and $f(x) = 0$ otherwise.
b) $\mathcal{F}f$, where f is integrable and piecewise smooth.
c) $\mathcal{F}(\mathcal{F}(\mathcal{F}f))$, where f is as in b).
d) $f_a(x) = f(ax)$, $a \neq 0$, where f is integrable.
e) $f_a(x) = \frac{1}{\sqrt{2\pi}} \frac{a-ix}{a^2+x^2}$, $a > 0$. *Hint:* $f_1 = \mathcal{F}g$ for $g(x) = e^{-x}$ $(x > 0)$ and $g(x) = 0$ otherwise.
f) $f(x) = (1 - x^2)e^{-x^2}$.

Problem 40 *Schwartz functions*
Which of the following functions belong to the Schwartz space

$$\mathcal{S}(\mathbb{R}) = \{f \in C^\infty(\mathbb{R}) : x^m \partial_x^n f(x) \text{ is bounded for all } m, n \in \mathbb{N}_0\} \ ?$$

a) $\frac{1}{1+x^2}$, b) e^{-x^2}, c) $(1 + x^{17})e^{-\sqrt{1+x^2}}$, d) $e^{-|x|}$, e) $e^{-x^2} \sin(e^{x^2})$.

Problem 41 *Convolution of Gaussian functions*
a) Let $f(x) = e^{-ax^2}$, $g(x) = e^{-bx^2}$, $a, b > 0$. Compute the convolution $f * g$ using the Fourier transform.

b) Derive the solution $u(t, x) = \frac{1}{\sqrt{4kt+1}} e^{-\frac{kx^2}{4kt+1}}$ of the heat equation $u_t = u_{xx}$ with initial condition $u(0, x) = e^{-kx^2}$ $(t > 0,\ x \in \mathbb{R},\ k > 0)$ from the general convolution formula for the solution.

c) Let $u_g(t, x)$ be the solution to the heat equation $u_t = u_{xx}$ satisfying $u(0, x) = g(x)$ $(t > 0,\ x \in \mathbb{R})$. Show $u_f(t+s, x) = u_{u_f(s,x)}(t, x)$ using c1) part a) or c2) the uniqueness of solutions to the heat equation.

Problem 42 *Solving partial differential equations*
a) The partial differential equation

$$u_t = c^2 u_{xx} + k u_x, \quad u(0, x) = f(x) \quad (x \in \mathbb{R}, t > 0)$$

models the evolution of an initial temperature distribution f in the presence of transport, e.g. when the wind is blowing. Solve this equation using the Fourier transform for the initial condition $f(x) = e^{-x^2}$.

b) Solve the following equation on $(t, x) \in (0, \infty) \times \mathbb{R}$ for given $f \in \mathcal{S}(\mathbb{R})$:

$$u_t = t u_{xx}, \quad u(0, x) = f(x) \ .$$

Problem 43 *Dirichlet problem on the upper half plane*

a) Solve the Dirichlet problem for the Laplace equation on the upper half plane,

$$u_{xx} + u_{yy} = 0, \quad u(x,0) = f(x) \quad ((x,y) \in \mathbb{R} \times (0\infty)) ,$$

in the special case $f(x) = 50$ for $|x| < 1$ and $f(x) = 0$ otherwise.

b) Verify that if $f : \mathbb{R} \to \mathbb{R}$ is bounded and uniformly continuous, the Poisson integral $\frac{y}{\pi} \int_{-\infty}^{\infty} \frac{f(s)}{(x-s)^2+y^2}\, ds$ converges to $f(x)$ uniformly in x as $y \to 0^+$.

Hint: Proceed as in the case of the heat equation.

Problem 44

Review the material from the lectures, exercises and assignments to this course. Ask if anything remains unclear to you.

Problem 45 *challenge — Poisson's sum formula, Fourier inversion and sampling*

Let $f : \mathbb{R} \to \mathbb{R}$ continuously differentiable and $|f(x)|, |\partial_x f(x)| \leq \frac{M}{1+|x|^{1+\varepsilon}}$ for some $\varepsilon, M > 0$. You may use (or prove yourself using the criterion of Weierstrass) that $F(x) := \sum_{n=-\infty}^{\infty} f(x + 2\pi n)$ converges uniformly and defines a continuously differentiable 2π–periodic function.

a) Determine the Fourier coefficients $\hat{F}(n)$ of F.

b) Use part a) to prove *Poisson's sum formula*

$$\sum_{n=-\infty}^{\infty} f(x + 2\pi n) = \frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} e^{inx}\ \mathcal{F}f(n).$$

Applying b) to the functions $f_\lambda(x) = f(\lambda x)$, one obtains $f(0) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathcal{F}f(\omega)\, \mathrm{d}\omega$ for $\lambda \to \infty$ (assuming that $\mathcal{F}f$ is integrable). Note that the right hand side of Poisson's formula is just a Riemann sum which converges to an integral for $\lambda \to \infty$. Replacing f by $(\tau_y f)(x) = f(x+y)$, gives the representation theorem for the Fourier transform: $f(y) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{iy\omega}\ \mathcal{F}f(\omega)\mathrm{d}\omega$.

c) Assume that $\mathcal{F}f(\omega) = 0$ for $|\omega| > c$ (engineers would say that f *is band–limited*). Show that

$$f(x) = \sum_{n\in\mathbb{Z}} f\left(\frac{n\pi}{c}\right) \frac{\sin(cx - n\pi)}{cx - n\pi} ,$$

by applying b) to $\mathcal{F}f_\lambda$, $f_\lambda(x) = f(\lambda x)$ for suitable $\lambda > 0$. To engineers, this fact is known as *Shannon's sampling theorem.* It allows you to reconstruct f from its values at multiples of $\frac{\pi}{c}$. The smallest possible value of $\frac{c}{\pi}$ is called *Nyquist frequency.* How is this sampling theorem related to the size of your iPod?