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PDL — Exercises 7

non-rectangular domains, Bessel functions, maximum principle

Problem 36 *Bessel functions and the wave equation in polar coordinates*

Given $p \geq 0$, the ordinary differential equation

$$x^2y'' + xy' + (x^2 - p^2)y = 0$$

is named after Bessel. Being of second order and linear, the general solution of this equation is $y(x) = c_1J_p(x) + c_2Y_p(x)$ for two linearly independent solutions J_p, Y_p. While J_p is bounded for $x \to 0^+$, Y_p is not. Bessel's equation arises whenever one tries to use separation of variables in polar or cylindrical coordinates for equations involving the Laplace operator, e.g. the heat, wave or eigenvalue problems for the Laplace equation.

a) Check that for $p = \frac{1}{2}$ the general solution is given by $y(x) = c_1\frac{\sin(x)}{\sqrt{x}} + c_2\frac{\cos(x)}{\sqrt{x}}$.

b) Find a power series solution of Bessel's equation by making the ansatz $y(x) = \sum_{n=0}^{\infty} a_n x^{r+n}$, $a_0 \neq 0$: First, use Bessel's equation to express a_n in terms of a_{n-2} and use $a_0 \neq 0$ to show that $r = \pm p$. Which of these values for r gives rise to a solution that it bounded for $x \to 0$? For the bounded solution J_p, determine the coefficients a_n in terms of a_0.

c) Interchanging sum and integral, formally check that some antiderivative of $x^{p+1}J_p(x)$ is proportional to $x^{p+1}J_{p+1}(x)$. By convention, a_0 is defined so that $\int x^{p+1}J_p(x)\,dx = x^{p+1}J_{p+1}(x) + C$ holds.

d) Use the Bessel function expansions from the lecture to solve the wave equation $u_{tt} = c^2\Delta u$ on the disk $\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < a^2\}$ with boundary condition $u(t,x,y) = 0$ for $x^2 + y^2 = a^2$ and initial condition $u(0,x,y) = f(x,y)$, $u_t(0,x,y) = g(x,y)$ when
d1) $a = 1$, $c = 10$, $f(x,y) = 1 - x^2 - y^2$, $g(x,y) = 1$, d2) $a = 2$, $c = 1$, $f(x,y) = 0$, $g(x,y) = 1$.

Problem 37 *Laplace equation in polar coordinates*

a) Find the solution to the Laplace equation $\Delta u = 0$ on the unit disk $\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$ satisfying, in polar coordinates (r,θ), the boundary condition $u(1,\theta) = \pi - \theta$ $(0 \leq \theta \leq \pi)$ and $u(1,\theta) = 0$ otherwise.

b) Suppose that $u(r,\theta)$ is a solution to the Laplace equation $\Delta u = 0$ which is independent of the angle θ. Show that $u(r,\theta) = a\ln(r) + b$ for some $a, b \in \mathbb{R}$.

c) Solve the Laplace equation $\Delta u = 0$ in the wedge $W = \{(x,y) \in (0,\infty)^2 : x^2 + y^2 < 1,\ y < x\}$ subject to homogeneous Dirichlet boundary conditions on the subsets of the boundary where $y = 0$ or $y = x$ and the Neumann boundary condition $\partial_n u(x,y) = y$ on the boundary part where $x^2 + y^2 = 1$.

Hint: Express everything in polar coordinates! As always, ∂_n denotes the derivative in the direction of the outward pointing normal vector to the boundary. Why is it ∂_r on the part where $x^2 + y^2 = 1$?

Problem 38 *Weak maximum principle and uniqueness of solutions for the Laplace equation*

Let $U \subset \mathbb{R}^2$ be a bounded open subset and $A, B, C, D, E, f : \overline{U} \to \mathbb{R}$ continuous functions such that $AC - B^2 > 0$, $A \geq 0$. Assume that u is a twice continuously differentiable solution of the equation $Lu := Au_{xx} + 2Bu_{xy} + Cu_{yy} + Du_x + Eu_y = f$ on U, which is continuous on $\overline{U}$.

a) Show that the stated equation is elliptic. Which A, B, C, D, E correspond to the Laplace equation?

b) Suppose that u has a maximum in the point $(x_0, y_0) \in U$.
b1) Show that $u_x(x_0, y_0) = u_y(x_0, y_0) = 0$ and that the matrix $\begin{pmatrix} u_{xx}(x_0, y_0) & u_{xy}(x_0, y_0) \\ u_{xy}(x_0, y_0) & u_{yy}(x_0, y_0) \end{pmatrix}$ has only non–positive eigenvalues.
b2) Conclude $Lu(x_0, y_0) \leq 0$.

c) Assume that $Lu = f > 0$ in U. Show that u does not have a maximum in U.

d) If $Lu = f \geq 0$ in U, consider the function $u_\epsilon(x, y) = u(x, y) + \epsilon e^{\gamma x}$ with $\epsilon > 0$ and $\gamma^2 A + \gamma D \geq 0$.
d1) Show that u_ϵ does not have a maximum in U.
d2) Letting $\epsilon \to 0$, conclude that the maximum of u is achieved on the boundary ∂U.

e) Show that if $Lu = f \leq 0$, then the minimum of u is achieved on the boundary ∂U.

f) Let u_1, u_2 be two solutions as above of $Lu_{1/2} = f_{1/2}$. Show that if $f_1 \leq f_2$ on U and $u_1 \geq u_2$ on ∂U, then $u_1 \geq u_2$ on U. Conclude that the Laplace equation with Dirichlet boundary conditions admits a unique solution.