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October 5, 2010

PDL — Exercises **6**

solving the Laplace equation, energy arguments, non-Cartesian coordinates

Problem 31 *Laplace operator and eigenfunctions*

Determine the Fourier series solutions of the following Dirichlet problems:

a) $u_{xx} + u_{yy} = 0$, $u(x,0) = 100$, $u(x,1) = u(0,y) = 0$, $u(2,y) = 100(1-y)$ $((x,y) \in (0,2) \times (0,1))$.
b) $u_{xx} + u_{yy} = 100$, $u(x,0) = 100$, $u(x,1) = u(0,y) = 0$, $u(2,y) = 100(1-y)$ $((x,y) \in (0,2) \times (0,1))$.

Discuss the convergence of the series you obtain in part a)!

Problem 32 *Eigenfunction expansions for the heat equation*

Solve the initial value problem for the two–dimensional heat equation $u_t = u_{xx} + u_{yy}$, $u(0,x,y) = f(x,y)$, $u(t,0,y) = u(t,a,y) = u(t,x,0) = u(t,x,b) = 0$ for $t > 0$, $(x,y) \in (0,a) \times (0,b)$ by reducing it to an eigenfunction problem.

Hint: The coefficients in the eigenfunction expansion should be functions of t.

Problem 33 *Energy arguments in higher dimensions: uniqueness and finite speed of propagation*

Let $U \subset \mathbb{R}^2$ be a bounded open set with piecewise smooth boundary ∂U, $\Delta = \partial_x^2 + \partial_y^2$. Consider the wave equation

$$u_{tt} - \Delta u = h \ , \quad u(0,x,y) = f(x,y) \ , \quad u_t(0,x,y) = g(x,y) \qquad (t \geq 0, (x,y) \in U)$$

with $u(t,x,y) = d(x,y)$ for $(x,y) \in \partial U$.

a) If u is a classical solution to this problem with $d = h = 0$, show that

$$E(t) = \frac{1}{2} \int_U u_t(t,x,y)^2 \ dx + \frac{1}{2} \int_U \left\{ u_x(t,x,y)^2 + u_y(t,x,y)^2 \right\} \ dx \ dy$$

satisfies $E(t) = E(0)$ for all t.

Hint: You have seen this before in one dimension. Verify that your proof still works! Can you do it in n dimensions?

b) Conclude that if u_1, u_2 are two classical solutions, then the energy associated to $u_1 - u_2$ is 0 for all t. Why does this imply that the wave equation has a unique solution?

For simplicity, let now $U = \mathbb{R}^2$. Given $(x_0, y_0) \in \mathbb{R}^2$, $t_0 > 0$ and $\tau \in [0, t_0]$, consider the balls $B(\tau) = \{(0,x,y) \in \{0\} \times U : (x-x_0)^2 + (y-y_0)^2 \leq (t_0 - \tau)^2\}$ and the cone $C = \{(t,x,y) \in [0,t_0] \times U : (x-x_0)^2 + (y-y_0)^2 \leq (t_0 - t)^2\}$ in (t,x,y)–space. Define the local energy

$$e(t) = \frac{1}{2} \int_{B(t)} u_t(t,x,y)^2 \ dx \ dy + \frac{1}{2} \int_{B(t)} \left\{ u_x(t,x,y)^2 + u_y(t,x,y)^2 \right\} \ dx \ dy \quad (0 \leq t \leq t_0) \ .$$

c) Sketch B and C.

d) Let $\partial_n u$ be the directional derivative in the direction of the outward unit normal vector to $\partial B(t)$. Show

$$\partial_t e(t) = \int_{\partial B(t)} \left\{ (\partial_n u)\ u_t\ - \frac{1}{2} u_t^2 - \frac{1}{2}\left(u_x^2 + u_y^2\right) \right\}\ dS\ .$$

e) Noting that $|(\partial_n u)\ u_t| \leq |u_t| \sqrt{u_x^2 + u_y^2} \leq \frac{1}{2} u_t^2 + \frac{1}{2}\left(u_x^2 + u_y^2\right)$, show $\partial_t e(t) \leq 0$.

f) Conclude that if $u = u_t = 0$ in $B(0)$, then $u = 0$ in C. How does this relate to the "intervals of dependence" in one dimension?

Problem 34 *different coordinate systems*

a) Consider a function $u(r, \phi, \theta)$ in spherical coordinates. Compute Δu if u is independent of:
a1) ϕ and θ, a2) θ, a3) r .

b) Let $f : \mathbb{R}^n \to \mathbb{R}$ depend only on the radial variable $r = \sqrt{x_1^2 + \cdots + x_n^2}$: $f(x_1, \ldots, x_n) = \Phi(\sqrt{x_1^2 + \cdots + x_n^2})$. Show that $\Delta f := \sum_{i=1}^n \partial_{x_i}^2 f = \partial_r^2 \Phi + \frac{n-1}{r} \partial_r \Phi$.

Problem 35 *challenge — counting eigenfunctions*

The eigenfunctions of the Laplace operator on the square $0 < x, y < a$, which vanish on the boundary, are given by $\phi_{mn}(x, y) = \sin(\frac{m\pi x}{a}) \sin(\frac{n\pi y}{a})$, $m, n \in \mathbb{N}$. The corresponding eigenvalues are $\lambda_{mn} = -\frac{\pi^2}{a^2}\{m^2 + n^2\}$, i.e. $\partial_x^2 \phi_{mn} + \partial_y^2 \phi_{mn} = \lambda_{mn} \phi_{mn}$. Given $\lambda > 0$, let $N(\lambda)$ be the number of eigenfunctions ϕ_{mn} associated to eigenvalues satisfying $|\lambda_{mn}| < \lambda$. Show that $\lim_{\lambda \to \infty} \frac{N(\lambda)}{\lambda} < \infty$.

Remark: This is called Weyl's law *and holds in a slightly adapted form for any elliptic operator. Physicists have several names for it, e.g. for the radiation of a black body it is called* Rayleigh–Jeans law.