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PDL — Exercises **5**

convergence questions, solving the 1- and 2-dimensional heat equations with various boundary conditions

Problem 28 *Classical solutions of the wave equation*

Consider the wave equation

$$u_{tt} = u_{xx}\ ,\ u(t,0) = u(t,L) = 0\ ,\ u(0,x) = x^k(1-x)^k,\ u_t(0,x) = 0 \quad (0 < x < 1\ ,\ t \in \mathbb{R}\ ,\ k \in \mathbb{N})\ .$$

For which values of k is the solution classical? Answer the question using first d'Alembert's solution and then the representation of the solution as a Fourier series. What happens if we replace the initial conditions by $u(0,x) = 0$, $u_t(0,x) = x^k(1-x)^k$?

Problem 29 *Various boundary conditions*

a) Solve the nonhomogeneous boundary value problem for the heat equation

$$u_t(t,x) = u_{xx}(t,x)\ ,\quad u(t,0) = 100,\quad u(t,1) = 100\ ,\quad u(0,x) = 50x(1-x)\quad (0 < x < 1\ ,\ t > 0).$$

Does the Fourier series converge for $t < 0$?

b) Using the ideas behind the solution of the nonhomogeneous boundary value problem for the heat equation, solve the corresponding problem

$$u_{tt}(t,x) = c^2 u_{xx}(t,x)\ ,\quad u(t,0) = A,\quad u(t,L) = B\ ,\quad u(0,x) = f(x)\ ,\quad u_t(0,x) = g(x)\quad (0 < x < L\ ,\ t > 0)$$

for the wave equation.

Hint: You should replace the steady state solution from the case of the heat equation by $v(x) = \frac{B-A}{L}x + A$.

c) Use separation of variables and Fourier series to solve the heat equation with insulated ends,

$$u_t(t,x) = u_{xx}(t,x)\ ,\quad u_x(t,0) = u_x(t,L) = 0\ ,\quad u(0,x) = f(x)\quad (0 < x < L\ ,\ t > 0).$$

d) For the heat equation with a radiating end, we encountered generalized Fourier series $\sum_{n=1}^{\infty} c_n \sin(\mu_n x)$, where the μ_n are the positive solutions of $\tan(\mu L) = -\frac{\mu}{\kappa}$. For $m \neq n$, verify the orthogonality relation

$$\int_0^L \sin(\mu_m x)\ \sin(\mu_n x)\ dx = \frac{1}{\mu_n^2 - \mu_m^2}\left\{\mu_m \sin(\mu_n L)\cos(\mu_m L) - \mu_n \cos(\mu_n L)\sin(\mu_m L)\right\} = 0\ .$$

Problem 30 *2–dimensional heat equation*

Find the Fourier series solution of the two–dimensional heat equation

$$u_t(t,x,y) = u_{xx}(t,x,y) + u_{yy}(t,x,y)\ ,\quad u(t,x,0) = u(t,x,1) = u(t,0,y) = u(t,1,y) = 0\ ,$$
$$u(0,x,y) = x(1-x)y(1-y)\quad (0 < x,y < 1\ ,\ t > 0).$$