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PDL — Exercises 4

linear PDE, solving the wave equation: Fourier series and d'Alembert's solution

Problem 22 *Terminology*

a) Which of the following equations in $\mathbb{R}^2$ are scalar, linear or (if linear) homogeneous? What is their order? For the scalar, linear ones of second order, determine whether they are elliptic, parabolic or hyperbolic.

a1) $\alpha u_t u_x + u_{xt} = 2u,\ u(t,0) + u_x(t,0) = 0\ (\alpha \in \mathbb{R})$, a2) $u_{xx} + e^t u_{tt} = u\cos(x),\ u(0,x) + u(1,x) = 0$,
a3) $u_{xx} + 2u_{xt} + u_{tt} + 2u_x + 2u_t + u = 0$, a4) $u_t - u_x = v,\ v_t + v_x = 0,\ u(0,x) = 0,\ v(0,x) = 3$.

b) Show that the function $u(t,x) = e^{ax}e^{bt}$, $a, b \in \mathbb{C}$, is a solution of a3) if and only if a and b satisfy $a^2 + 2ab + b^2 + 2a + 2b + 1 = 0$. Find at least four linearly independent solutions of a3)!

Problem 23 *Solving the wave equation*

Consider the wave equation $u_{tt} = c^2 u_{xx},\ t > 0,\ x \in (0,1)$, with homogeneous boundary conditions $u(t,0) = u(t,1) = 0$ for $t > 0$ subject to the following initial conditions:

α) $u(0,x) = \sin(\pi x)\cos(\pi x)$, $u_t(0,x) = 0$ for $x \in (0,1)$, $c = \frac{1}{\pi}$,
β) $u(0,x) = 4x$ for $x \in (0, \frac{1}{4}]$, $u(0,x) = 1$ for $x \in (\frac{1}{4}, \frac{3}{4})$, $u(0,x) = 4(1-x)$ for $x \in [\frac{3}{4}, 1)$,
$u_t(0,x) = 1$ for $x \in (0,1)$, $c = 4$.

a) Solve the equations in two ways, using first Fourier series and then d'Alembert's solution.

b) Sketch the corresponding motions of a string graphically.

c) For problem α), find the characteristic lines of the equation in (t,x)–space. What is the interval of dependence of the point $(t_0, x_0) = (0.2, 0.5)$? For which (t,x) can you use intervals of dependence to compute the solution in terms of the initial conditions alone? I.e. determine all (t,x) such that $u(t,x)$ does not depend on the imposed boundary conditions. Compute u for these (t,x)! Use characteristic parallelograms to determine $u(5, 0.2)$.

Problem 24 *Time period of motion*

a) Show that the n-th normal mode

$$u_n(t,x) = \sin\left(\frac{n\pi x}{L}\right)\left(b_n \cos\left(\frac{cn\pi t}{L}\right) + b_n^* \sin\left(\frac{cn\pi t}{L}\right)\right)$$

is $\frac{2L}{nc}$–periodic in t.

b) Conclude that any superposition of normal modes is $\frac{2L}{c}$–periodic and that the string thus vibrates with a time period $\frac{2L}{c}$.

Problem 25 *Fourier series and a damped wave equation*

Use Fourier series to solve the following equation for $(t,x) \in (0,\infty) \times (0,\pi)$:

$$u_{tt} + u_t = u_{xx}, \quad u(t,0) = u(t,\pi) = 0, \quad u(0,x) = \sin(x), \quad u_t(0,x) = 0\ .$$

Problem 26 *Conservation of energy*

The energy at time t of a vibrating string fixed at $x = 0$ and $x = L$ is given by $E(t) = \frac{1}{2}\int_0^L \left\{u_t^2(t,x) + c^2 u_x^2(t,x)\right\}\,dx$.

a) Differentiate under the integral sign to show that

$$\partial_t E = \int_0^L \left\{u_t u_{tt} + c^2 u_x u_{tx}\right\}\,dx = c^2 \int_0^L (u_x u_t)_x\,dx$$

for any twice continuously differentiable solution of the wave equation.

b) Using the boundary conditions, show that $u_t(t,0) = u_t(t,L) = 0$ for all $t > 0$ and conclude $E(t) = E(0)$ for all $t > 0$.

Problem 27 *Generalized solutions*

A function $u(t,x)$ is said to be a weak solution of the wave equation, if

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u(t,x)\{\phi_{tt}(t,x) - c^2\phi_{xx}(t,x)\}\,dt\,dx = 0$$

for all twice continuously differentiable functions $\phi : \mathbb{R}^2 \to \mathbb{R}$ which are compactly supported (i.e. equal to 0 for large $|t| + |x|$).

a) Convince yourself that the weak solutions of the wave equation form a vector space.

b) Show that every twice continuously differentiable solution of the wave equation is a weak solution (integrate by parts!).

c) Let $H(\alpha) = 1$ if $\alpha \geq 0$ and $H(\alpha) = 0$ otherwise. Verify that the discontinuous functions $H(ct - x)$ and $H(ct + x)$ are weak solutions of the wave equation. What about $f(ct - x) + g(ct + x)$ for arbitrary piecewise continuous f, g?