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PDL — Exercises 3

Fourier series

Problem 17 *Fourier series, mean–square approximation and half–range expansions*

a) Let $p > 0$. Compute the Fourier series of the $2p$–periodic functions defined by

a1) $f(x) = a\left(1 - \frac{x^2}{p^2}\right)$ for $x \in [-p, p)$, $a \in \mathbb{R}$, a2) $f(x) = x^2$ for $x \in [-p, p)$,

a3) $f(x) = \sinh(ax)$ for $x \in [-p, p)$, $p = \pi$, $a \notin i\mathbb{Z}$.

Express your answers both as real (involving cos and sin) and as complex (involving $e^{in\pi x/p}$) series.

b) Let $s_N(x) = a_0 + \sum_{n=1}^N \left\{a_n \cos\left(\frac{n\pi x}{p}\right) + b_n \sin\left(\frac{n\pi x}{p}\right)\right\}$ be the N-th partial sum of the Fourier series of the function f in a1). Determine $E_N = \frac{1}{2p}\int_{-p}^{p}(f(x) - s_N(x))^2\, dx$ for $p = \pi$ and $N = 1, 2, 3$.

c) Use the result of a2) to prove $\sum_{n=1}^\infty \frac{(-1)^n}{n^2} = -\frac{\pi^2}{12}$ and $\sum_{n=1}^\infty \frac{1}{n^2} = \frac{\pi^2}{6}$.

d) Determine the cosine and sine series expansions of the function $f(x) = x^2$ for $x \in (0, p)$.

e) For $x \notin i\pi\mathbb{Z}$ obtain the expansion $\coth(x) = \sum_{n=-\infty}^\infty \frac{x}{x^2+(n\pi)^2}$ from the example of a complex Fourier series which was presented in the lecture.

Problem 18 *Parseval's identity and uniform convergence*

a) Compute $\int_{-\pi}^{\pi}\left(1 + \sum_{n=1}^\infty 3^{-n}\cos(nx) + \frac{1}{n}\sin(nx)\right)^2\, dx$ and $\int_{-\frac{1}{2}}^{\frac{1}{2}}\left(\sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{(-1)^n}{n} e^{2\pi i n x}\right)^2\, dx$.

b) Given a 2π–periodic function with Fourier coefficients $a_0 = 0$, $a_n = \frac{(-1)^n}{n^2}$ and $b_n = \frac{1}{n}$ $(n \in \mathbb{N})$, is it continuous? What if $a_n = \frac{1}{1+n^2}$ and $b_n = \frac{1}{n^3}$ $(n \in \mathbb{N})$ instead?

c) Which of the Fourier series in 17a) are uniformly convergent? Answer without looking at the Fourier coefficients!

d) Is there a function $f \in L^2(-\pi, \pi)$ such that $\int_{-\pi}^{\pi} f(x)\, e^{ik^3 x}\, dx = \frac{1}{k}$?

Problem 19 *Differentiation of Fourier series*

Suppose that f is a $2p$–periodic piecewise smooth, continuous function such that f' is also piecewise smooth. Let a_n, b_n denote the Fourier coefficients of f and a'_n, b'_n those of f'.

a) Show that (integration by parts!)

$$a'_0 = 0,\ a'_n = \frac{n\pi}{p}\, b_n,\ b'_n = -\frac{n\pi}{p}\, a_n\ .$$

b) Conclude that the Fourier series of f' is obtained from that of f by differentiating term by term. I.e. if

$$a_0 + \sum_{n=1}^{\infty}\left\{a_n \cos\left(\frac{n\pi x}{p}\right) + b_n \sin\left(\frac{n\pi x}{p}\right)\right\}$$

is the Fourier series of f, then the Fourier series of f' is given by

$$\sum_{n=1}^{\infty}\left\{-a_n \frac{n\pi}{p}\sin\left(\frac{n\pi x}{p}\right) + b_n\frac{n\pi}{p}\cos\left(\frac{n\pi x}{p}\right)\right\}\ .$$

c) If the complex Fourier series of f is $\sum_{n\in\mathbb{Z}} c_n e^{i\frac{n\pi x}{p}}$, what is the complex Fourier series of f'?

d) Show that for all $x \in \mathbb{R}$, $\lim_{n\to\infty} \cos(nx) \neq 0$. Conclude that $\sum_{n=1}^{\infty} \cos(nx)$ is divergent for all x.

Hint: If $\lim_{n\to\infty} \cos(nx) = 0$, then also $\lim_{n\to\infty} \cos^2(nx) = 0$ and $\lim_{n\to\infty} \cos(2nx) = 0$. But this contradicts $\cos^2(nx) = \frac{1}{2}(1 + \cos(2nx))$.

e) Consider the Fourier series of the sawtooth function $\sum_{n=1}^{\infty} \frac{\sin(nx)}{n}$. Show that the function represented by this series satisfies all the hypotheses of part b), except that it fails to be continuous. Use d) to show that this series cannot be differentiated term by term.

Problem 20 *Pointwise convergence theorem*

In the lecture the following two facts were stated, but not demonstrated, in the proof of the pointwise convergence theorem:

a) Properties of $D_N(\alpha) = \frac{\sin((N+\frac{1}{2})\alpha)}{\sin(\frac{1}{2}\alpha)} = 1 + 2\sum_{n=1}^{N} \cos(n\alpha)$:

$D_N(\alpha) = D_N(-\alpha)$, $\frac{1}{\pi}\int_{-\pi}^{\pi} D_N(x)\, dx = 1$.

b) Riemann–Lebesgue Lemma: If h is piecewise continuous, then

$$\lim_{\alpha\to\infty} \int_{-\pi}^{\pi} h(x)\ \cos(\alpha x)\ dx = \lim_{\alpha\to\infty} \int_{-\pi}^{\pi} h(x)\ \sin(\alpha x)\ dx = 0\ .$$

Prove these assertions or look them up in the literature!

Problem 21 *challenge — Gibbs phenomenon*

Consider the 2π–periodic sawtooth function given by $s(x) = \pi - x$ for $x \in [0, 2\pi)$. We have previously seen that it has the Fourier series $\sum_{n=1}^{\infty} \frac{1}{n}\ \sin(nx)$. Consider the difference $g_N(x) = \sum_{n=1}^{N} \frac{1}{n}\ \sin(nx) - s(x)$, $x \in [0, 2\pi)$, between the N-th partial sum of the Fourier series and the function itself. In class, we saw (pictures showing) that close to 0, the convergence is not uniform: $s_N(x)$ develops a kind of spike overshooting its limiting values $s(x)$ and then oscillates for small $x > 0$. This exercise shows that the "spike" moves closer and closer to $x = 0$ as $N \to \infty$, but does not disappear.

a) Recall what we discussed as "Gibbs phenomenon" in class and summarize it in a picture.

b) Verify that $\partial_x g_N(x) = \frac{\sin((N+\frac{1}{2})x)}{\sin(\frac{1}{2}x)}$.

c) Why is $g_N(0) = -\pi$? Use the fundamental theorem of calculus to show $g_N(x) = \int_0^x \frac{\sin((N+\frac{1}{2})x)}{\sin(\frac{1}{2}x)}\ dx - \pi$.

d) Let x_N be the smallest $x > 0$ such that $\partial_x g_N(x) = 0$. Check that $x_N = \frac{\pi}{N+\frac{1}{2}}$.

e) Show that $\lim_{N\to\infty} g_N(x_N) > 0$. I.e. the height of the "spike" in the difference between the sawtooth function and the partial sums of its Fourier series does not go to zero.

f) Revise your sketch from a) and say more precisely what is happening.