Institut for Matematiske Fag
Københavns Universitet
Heiko Gimperlein, Kim Petersen

September 7, 2010

PDL — Exercises **2**

The method of characteristics, the wave equation and Fourier series

Problem 8 *Method of characteristics*

For nonzero constants a, b, consider the equations $a\partial_x u(x,y) + b\partial_y u(x,y) = 0$ and $e^{x^2}\partial_x u(x,y) + x\partial_y u(x,y) = 0$ $(x, y \in \mathbb{R})$.

a) What is each of the equations saying about the directional derivative of u?

b) Determine the characteristic curves in both cases.

c) Find all solutions of the equations using the method of characteristic curves.

d) Does either of the equations admit a solution with $u(0, y) = y$?

Problem 9 *Wave equation*

Let $c, L > 0$. In class, the vibrating string was described by the wave equation

$$\partial_t^2 u(t,x) = c^2 \partial_x^2 u(t,x) \qquad (t \in \mathbb{R},\ x \in (0,L))\ .$$

a) Let $\alpha \neq 0$ and $L(t,x) := (\alpha t, x)$ be a rescaling of the time axis. Show that α can be chosen so that $v(t,x) := u(L(t,x))$ satisfies the wave equation with $c = 1$.

b) Find the solution of the wave equation with $c = 1$ satisfying the boundary conditions $u(t,0) = u(t,L) = 0$ for all $t \in \mathbb{R}$ and either of the following initial conditions

b1) $u(0,x) = \frac{1}{2}\sin(\frac{\pi x}{L}) + \frac{1}{4}\sin(\frac{3\pi x}{L})$, $\partial_t u(0,x) = 0$, b2) $u(0,x) = 0$, $\partial_t u(0,x) = \sin(\frac{\pi x}{L})$.

For b2), also sketch the evolution of the string as a function of time and find the smallest $t_0 > 0$ with $\partial_t u(t_0, x) = 0$.

c) Express the solutions from part b) in terms of the normal modes $u_n(t,x) = \sin(\frac{n\pi x}{L})\sin(\frac{n\pi t}{L})$ and $v_n(t,x) = \sin(\frac{n\pi x}{L})\cos(\frac{n\pi t}{L})$.

Problem 10 *Piecewise smooth functions*

For which $n \in \mathbb{N}$ is the function $f : [-1,1] \to \mathbb{R}$ defined by $f(x) = x^n \sin(\frac{1}{x})$ for $x \neq 0$ and $f(0) = 0$ continuous, smooth, piecewise continuous or piecewise smooth on $[-1,1]$?

Problem 11 *Piecewise continuous periodic functions are bounded*

Let f be piecewise continuous and periodic. Show that f is bounded.

Hint: You can restrict to the interval $[0,T]$, where T is a period of f. Let $x_1, \ldots, x_n \in (0,T)$ be the points of discontinuity of f, $x_0 = 0$ and $x_{n+1} = T$. Define $f_j(x) = f(x)$ for $x \in (x_{j-1}, x_j)$ and extend it continuously to a function on $[x_{j-1}, x_j]$ by setting $f_j(x_{j-1}) = \lim_{x \to x_{j-1}^+} f(x)$ and $f_j(x_j) = \lim_{x \to x_j^-} f(x)$. Now, what do you know about continuous functions on closed, bounded intervals?

Problem 12 *Orthogonality of the trigonometric system*

Let $m, n \in \mathbb{N}$. Compute $\int_{-\pi}^{\pi} e^{inx}\, e^{-imx}\, dx$ and use the result to show that

$$\begin{aligned}
\int_{-\pi}^{\pi} \cos(mx)\, \sin(nx)\, dx &= 0 \;, \\
\int_{-\pi}^{\pi} \cos(mx)\, \cos(nx)\, dx &= 0 \quad (m \neq n) \;, \\
\int_{-\pi}^{\pi} \sin(mx)\, \sin(nx)\, dx &= 0 \quad (m \neq n) \;, \\
\int_{-\pi}^{\pi} \cos(nx)^2\, dx = \int_{-\pi}^{\pi} \sin(nx)^2\, dx &= \pi \quad (n \neq 0) \;, \\
\int_{-\pi}^{\pi} \cos(0x)^2\, dx = 2\pi \;, \; \int_{-\pi}^{\pi} \sin(0x)^2\, dx &= 0 \;.
\end{aligned}$$

Problem 13 *Sums of periodic functions*

a) Show that if $f_1, \ldots, f_n, \ldots$ are T–periodic functions, $a_1, \ldots, a_n, \ldots$ is a sequence of real numbers and $\sum_{n=1}^{\infty} a_n f_n(x)$ converges for all $x \in [0, T)$, then its limit is a T–periodic function.

b) Let $f(x) = \cos(x) + \cos(\pi x)$. Show that the equation $f(x) = 2$ has a unique solution. Conclude that the sum of periodic functions does not have to be periodic.

Problem 14 *Periods*

Are the following functions periodic? If so, determine their fundamental period:

a) $\cos(\frac{1}{2}x) + 3\sin(2x)$, b) $\arctan(x)$ c) $\frac{1}{2+\sin(2x)}$, d) $\int_0^x \cos^{2010}(\frac{22}{7}s)\, ds$,

e) $f(x) = 1$ for $x \in \mathbb{Q}$ and $f(x) = 0$ otherwise, f) (challenge) $\cos\left(\partial_x \sinh\left(\log\left(\sum_{n=-\infty}^{\infty} e^{-n^2+2xn}\right) - x^2\right)\right)$.

Problem 15 *Computing Fourier series*

Compute the Fourier series of the following 2π–periodic functions:

a) $f(x) = |\sin(x)|$, b) $f(x) = x$ if $x \in [-\pi, \pi)$,

Use the Fourier series of part b) to prove $\frac{\pi}{4} = \sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1}$.

Problem 16 *Fourier coefficients of translated and reflected functions*

Let f be 2π–periodic and $a \in \mathbb{R}$. Express the Fourier coefficients of $g(x) = f(-x)$ and $h(x) = f(x-a)$ in terms of the Fourier coefficients of f.