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PDL — Exercises **1**

Recap of ordinary differential equations and elementary calculus

Problem 1 Browse through what you have learned in previous courses about ordinary differential equations. In particular, answer the following questions:

a) Let $p_0, \dots, p_{n-1}, g$ be continuous functions on the interval $J \subset \mathbb{R}$. What does the general solution of the equation

$$y^{(n)} + p_{n-1}(x)y^{(n-1)} ... + p_1(x)y' + p_0(x)y = g(x) \qquad (x \in J)$$

look like? Given initial conditions $y(0) = y_0, \dots, y^{(n-1)}(0) = y_{n-1}$, is there going to be a solution? How do you find all solutions if $p_0, \dots, p_{n-1}$ are constant?

b) Let p, g be continuous functions on the interval $J \subset \mathbb{R}$. Write down an explicit formula (up to integrals) for the general solution of the equation

$$y' + p(x)y = g(x) \qquad (x \in J).$$

Use this formula to solve the initial value problem $y' + xy = x$, $y(0) = 0$.

Problem 2 Find the general solution of the following ordinary differential equations:

a) $y'' + 2y' + y = 0$, b) $y'' + y = 0$.

For each of the equations, also determine the solution satisfying $y(0) = 2$, $y'(0) = -1$.

Problem 3 Find the general solution of the following ordinary differential equations:

a) $y'' + 2y' + y = e^{-x}$, b) $y'' + y = \sin(x) + \cos(x)$.

Problem 4 a) Solve the initial value problem

$$(1 - x^2)y'' - 2xy' + 12y = 0 \quad (x \in \mathbb{R}), \ y(0) = 0, \ y'(0) = 1 \ .$$

b) Compute the first 10 terms in the Taylor expansion around $x = 0$ of two linearly independent solutions to $(1 - x^2)y'' - 2xy' + 12y = 0$.

Problem 5 *(challenge)*
Show that any solution to the ordinary differential equation $-2yy'' + y' = -1$ satisfying $y(0) = 1$, $y'(0) = 0$ is strictly monotonously increasing. If you know any existence theorem for nonlinear ordinary differential equations, argue that this particular problem admits a unique solution on an interval $[0, X)$ for some $X > 0$. (Actually, there exists a unique solution on $[0, \infty)$.)

Problem 6 This exercise derives the general solution of the one-dimensional wave equation

$$\partial_t^2 u(t,x) - \partial_x^2 u(t,x) = 0 \qquad (t, x \in \mathbb{R})$$

with the help of a clever *change of variables*. The equation describes e.g. (approximately) the electric current $u(t,x)$ flowing through a point x at time t in a very thin, ideal wire or a water wave with watersurface described by $u(t,x)$ in a narrow and deep canal.
All functions in this exercise are assumed to be twice differentiable.

a) Show that $L(t,x) := (t+x, t-x)$ defines an invertible linear transformation $L : \mathbb{R}^2 \to \mathbb{R}^2$ and that L and L^{-1} are infinitely often differentiable (linear algebra!).

b) Given $u : \mathbb{R}^2 \to \mathbb{R}$, find $H : \mathbb{R}^2 \to \mathbb{R}$ such that $u(t,x) = H(L(t,x))$.

c) For this H, verify that

$$\partial_t^2 u(t,x) - \partial_x^2 u(t,x) = C \, (\partial_w \partial_z H(w,z))|_{(w,z)=L(t,x)}$$

for some $C \neq 0$ (chain rule!).

d) Find the general solution of $\partial_w \partial_z H(w,z) = 0$ (elementary calculus!).

e) Conclude that the general solution of $\partial_t^2 u(t,x) - \partial_x^2 u(t,x) = 0$ is given by $u(t,x) = g(t+x) + h(t-x)$, $g, h : \mathbb{R} \to \mathbb{R}$.

f) Given $f : \mathbb{R} \to \mathbb{R}$, find a solution of the initial value problem

$$\partial_t^2 u(t,x) - \partial_x^2 u(t,x) = 0 \quad (t, x \in \mathbb{R}), \;\; u(0,x) = f(x), \;\; \partial_t u(0,x) = 0 \qquad (x \in \mathbb{R}).$$

Interpret the result!