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# PDL — Assignment **3**

to be handed in by October 22, 2010, noon

*Please note that the final exam is going to be on Wednesday, November 3, 2010, 10 – 12 am in A110.*

**Problem A 11** *The Laplace operator on a rectangle*

a) Solve $u_{xx} + u_{yy} = 0$ , $u(x,0) = \sin(13\pi x)$ , $u(x,1) = u(0,y) = u(1,y) = 0 \quad ((x,y) \in (0,1) \times (0,1))$.

b) Use eigenfunction expansions to solve

$$u_{xx} + u_{yy} = 3u - 1 \ , \quad u(x,0) = u(x,1) = u(0,y) = u(1,y) = 0 \quad ((x,y) \in (0,1) \times (0,1)).$$

**Problem A 12** *Wave equation in a ball*

Let $\alpha_3$ be the third positive zero of the Bessel function $J_0$. Solve the wave equation $u_{tt} = u_{xx} + u_{yy}$ on the unit ball $\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$ with boundary condition $u(t,x,y) = 0$ for $x^2 + y^2 = 1$ and initial conditions $u(0,x,y) = J_0(\alpha_3\sqrt{x^2+y^2}) + 1 - x^2 - y^2$, $u_t(0,x,y) = 0$.

*Hint:* You may cite intermediate results from non-mandatory exercises or Asmar's book.

**Problem A 13** *Laplace equation in a ball I*

Consider the Laplace equation $\Delta u = u_{xx} + u_{yy} = 0$ in the unit ball $B_1 = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$ with boundary condition $u(x,y) = f(x,y)$ for $x^2 + y^2 = 1$.

a) Find the Fourier series solution in the case $f(x,y) = 100$ for $0 < y < x$ and $f(x,y) = 0$ otherwise.

b) Does your solution in a) satisfy the boundary condition in all points? Does it converge uniformly in $B_1$? Show that for every $c < 1$ the series converges uniformly in $\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < c\}$ and defines an infinitely often differentiable function there.

*Hint:* In the lecture we discussed similar questions on the rectangle for general boundary values.

**Problem A 14** *Laplace equation in a ball II*

Consider again the Laplace equation $\Delta u = 0$ in the unit ball $\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$ with boundary condition given in polar coordinates by $u(r,\theta) = f(\theta)$ for $r = \sqrt{x^2+y^2} = 1$. Assume that $f$ is continuous and piecewise smooth. You know from the lecture that in polar coordinates the solution has the Fourier series form

$$u(r,\theta) = a_0 + \sum_{n=1}^{\infty} r^n \left\{a_n \cos(n\theta) + b_n \sin(n\theta)\right\} \ .$$

a) Substituting the definitions of $a_n$ and $b_n$, show that $u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} f(\phi)\,(1 + 2\sum_{n=1}^{\infty} r^n \cos(n(\phi-\theta)))\ d\phi$.

b) Prove $1 + 2\sum_{n=1}^{\infty} r^n \cos(n(\phi-\theta)) = \frac{1-r^2}{1+r^2-2r\cos(\phi-\theta)}$ to obtain *Poisson's integral formula*

$$u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} f(\phi)\ \frac{1-r^2}{1+r^2-2r\cos(\phi-\theta)}\ d\phi \ .$$

*Hint:* $\sum_{n=1}^{\infty} r^n \cos(n\alpha) = -1 + \mathrm{Re}\ \sum_{n=0}^{\infty}(re^{i\alpha})^n = -1 + \mathrm{Re}\ \frac{1}{1-re^{i\alpha}}$, the real part of a geometric series!

**Problem A 15** *More on the Laplace equation*

a) Let $f : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$, $n > 2$, be a solution to the Laplace equation $\Delta u = \sum_{i=1}^n \partial_{x_i}^2 u = 0$ on $\mathbb{R}^n \setminus \{0\}$ which only depends on the radial variable $r = \sqrt{x_1^2 + \cdots + x_n^2} : u(x_1, \ldots, x_n) = \Phi(\sqrt{x_1^2 + \cdots x_n^2})$. Verify that $\Phi(r) = ar^{2-n} + b$ for some $a, b \in \mathbb{R}$.

*Hint:* Exercise 34b). Euler's differential equation was also solved in class, or see Asmar, Appendix A.3.

b) Solve the Laplace equation $\Delta u = 0$ in the quadrant $\{(x, y) \in (0, \infty)^2 : x^2 + y^2 < 1\}$ subject to homogeneous Dirichlet boundary conditions on the subsets of the boundary where $x = 0$ or $y = 0$ and Dirichlet boundary condition $u(x, y) = 2xy$ on the boundary part where $x^2 + y^2 = 1$.

*Hint:* Introduce polar coordinates! Simplify $2xy$. Can you even *guess* the (simple) solution?