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PDL — Assignment 2

to be handed in by October 8, 2010, noon

Problem A 6 *Solving the wave equation*

Consider the wave equation $u_{tt} = u_{xx}$, $t > 0$, $x \in (0,1)$, with homogeneous boundary conditions $u(t,0) = u(t,1) = 0$ for $t > 0$ subject to the initial condition $u(0,x) = 2x$ for $0 < x \leq \frac{1}{2}$ and $u(0,x) = 2(1-x)$ for $\frac{1}{2} < x < 1$, $u_t(0,x) = 0$ for $x \in (0,1)$.

a) Solve the equation in two ways, using first Fourier series and then d'Alembert's solution.

b) Does your solution really satisfy the wave equation for all $(t,x) \in \mathbb{R} \times (0,L)$?

c) Find the characteristic lines of the equation in (t,x)–space. What is the interval of dependence of the point $(t_0,x_0) = (0.2, 0.5)$? For which (t,x) can you use intervals of dependence to compute the solution in terms of the initial conditions alone? I.e. determine all (t,x) such that $u(t,x)$ does not depend on the imposed boundary conditions.

Problem A 7 *Periodic solutions for the wave equation*

Use d'Alembert's solution for the wave equation

$$u_{tt} = c^2 u_{xx}\ ,\ u(t,0) = u(t,L) = 0\ ,\ u(0,x) = f(x),\ u_t(0,x) = 0 \quad (0 < x < L\ ,\ t \in \mathbb{R})$$

to show that u is $\frac{2L}{c}$–periodic.

Problem A 8 *Energy in the heat equation*

The energy at time t of a solution to the heat equation

$$u_t(t,x) = u_{xx}(t,x)\ , \quad u(t,0) = u(t,L) = 0\ , \quad u(0,x) = f(x) \quad (0 < x < L\ ,\ t > 0)$$

is defined as $E(t) = \frac{1}{2}\int_0^L u^2(t,x)\ dx$.

a) Show that $\int_0^L \int_0^t \partial_s \left\{\frac{1}{2}u^2(s,x)\right\}\ ds\ dx = -\int_0^L \int_0^t (\partial_x u(s,x))^2\ ds\ dx$.

b) Deduce $E(t) \leq E(0)$ for all $t \geq 0$.

Problem A 9 *Solving the heat equation*

a) Find the Fourier series solution to the heat equation

$$u_t(t,x) = u_{xx}(t,x)\ , \quad u(t,0) = u(t,1) = 0\ , \quad u(0,x) = x(1-x) \quad (0 < x < 1\ ,\ t > 0).$$

b) Discuss the convergence of the Fourier series solution obtained in a). Does it also converge for $t < 0$?

c) Use your solution from a) to solve the nonhomogeneous boundary value problem

$$u_t(t,x) = u_{xx}(t,x)\ , \quad u(t,0) = 1, \quad u(t,1) = 0\ , \quad u(0,x) = (x+1)(1-x) \quad (0 < x < 1\ ,\ t > 0).$$

Problem A 10 *2–dimensional heat equation*

Find the Fourier series of the solution to the two–dimensional heat equation

$$u_t(t,x,y) = u_{xx}(t,x,y) + u_{yy}(t,x,y)\ , \quad u(t,x,0) = u(t,x,1) = u(t,0,y) = u(t,1,y) = 0\ ,$$
$$u(0,x,y) = \sin(\pi x)\sin(\pi y) \quad (0 < x,y < 1\ ,\ t > 0).$$