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PDL — Assignment 1

to be handed in by September 24, 2010, noon

*Problems with an asterisk * might be slightly more difficult than the others and are not mandatory.*

Problem A 1 *Method of characteristics*

Consider the equation $x\partial_x u(x,y) + y\partial_y u(x,y) = 0$, $x, y \in \mathbb{R}$.

a) What does it say about the directional derivative of $u(x,y)$?

b) Determine the characteristic curves.

c) Find all solutions using the method of characteristic curves.

Problem A 2 *Antiderivatives of periodic functions*

Suppose that $f : \mathbb{R} \to \mathbb{R}$ is piecewise smooth and T–periodic, and let $F(x) = \int_0^x f(t)\, dt$.

a) Show that F is T–periodic if and only if $\int_0^T f(t)\, dt = 0$.

b*) Show that F is piecewise smooth.

c) Show that if F is $T = 2p$–periodic, its Fourier series is given by

$$\frac{p}{\pi}\sum_{n=1}^{\infty}\frac{b_n}{n} + \sum_{n=1}^{\infty}\left\{-\frac{p}{n\pi}b_n \cos\left(\frac{n\pi x}{p}\right) + \frac{p}{n\pi}a_n \sin\left(\frac{n\pi x}{p}\right)\right\} ,$$

where a_n, b_n are the Fourier coefficients of f.

Hint: You may use the results of other exercises, in particular problems 11 and 19. If you do not know how to do part b), just assume that it is true and continue with part c).

Problem A 3 *Fourier series I*

Let f be the 2π–periodic function defined by $f(x) = \pi^2 x - x^3$ for $x \in (-\pi, \pi]$.

a) Compute the Fourier series $a_0 + \sum\{a_n \cos(nx) + b_n \sin(nx)\}$ of f.

b) Use the result from part a) to find the complex form of the Fourier series $\sum c_n e^{inx}$ of f.

c) Use Parseval's identity and the Fourier series of f to compute $\sum_{n=1}^{\infty} \frac{1}{n^6}$.

Problem A 4 *Fourier series II*

Let $0 < c < p$ and $f : \mathbb{R} \to \mathbb{R}$ the $2p$–periodic function defined by $f(x) = 1$ for $|x| < c$, $f(x) = \frac{p-|x|}{p-c}$ for $c \leq |x| \leq p$.

a) Compute the Fourier series of f.

b) Does it converge pointwise for all $x \in \mathbb{R}$? Does it converge uniformly?

c) Determine the cosine series expansion of the function $g : (0, p) \to \mathbb{R}$ defined by $g(x) = 1$ for $x \in (0, c)$, $g(x) = \frac{p-x}{p-c}$ for $x \in [c, p)$.

Problem A 5 *Decay of Fourier coefficients*

If f is a 2π–periodic k–times continuously differentiable function ($k \geq 1$) with Fourier coefficients a_n, b_n, then $\sum_{n=1}^{\infty} n^{2k}\{a_n^2 + b_n^2\} < \infty$.

Hint: Bessel's inequality. You may use without proof the relation between the Fourier coefficients of f and $\partial_x f$ from problem 19 even if $\partial_x f$ is merely continuous.