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We here supply the explanations in [A04] (short for Nakhlé Asmar: "Partial Differential Equations, with Fourier Series and Boundary Conditions", 2004) with some further information.

1. Convergence of Fourier Series and applications

1.1 Convergence questions for the one-dimensional wave equation.

The most important classes of functions considered in Chapter 2 are the periodic functions that are piecewise C^1 ([A04] says "piecewise smooth"), and those that are piecewise C^1 *and* continuous. We recall that a function f on an interval $[a,b]$ is piecewise C^1 when there are points $t_0 = a < t_1 < t_2 < \dots < t_m < t_{m+1} = b$ such that f is C^1 on each of the open intervals $]t_i, t_{i+1}[$, with f and f' having limits at the endpoints.

Let us introduce the abbreviation PC1C for "piecewise C^1 and continuous".

Recall from [A04, Sections 2.2–3] that when $f : \mathbb{R} \to \mathbb{R}$ has period $2L$ and is piecewise continuous, then it has a Fourier series

$$a_0 + \sum_{n=1}^{\infty} \big(a_n \cos(\frac{n\pi}{L}x) + b_n \sin(\frac{n\pi}{L}x)\big), \tag{1.1}$$

where the coefficients are determined from f by:

$$\begin{aligned} a_0 &= \frac{1}{2L}\int_{-L}^{L} f(x)\,dx, \\ a_n &= \frac{1}{L}\int_{-L}^{L} f(x)\cos(\frac{n\pi}{L}x)\,dx \text{ for } n \ge 1, \\ b_n &= \frac{1}{L}\int_{-L}^{L} f(x)\sin(\frac{n\pi}{L}x)\,dx \text{ for } n \ge 1. \end{aligned} \tag{1.2}$$

(It is sufficient for this that f^2 is integrable on $[-L, L]$, but the book is not very explicit on this point.)

One can show (see Section 2.8):

Theorem 1.1. *If $f : \mathbb{R} \to \mathbb{R}$ has period $2L$ and is piecewise C^1, then its Fourier series (1.1) converges pointwise, in the sense that for each $x \in \mathbb{R}$, the partial sum $s_N(x) = a_0 + \sum_{n=1}^{N} \big(a_n \cos(\frac{n\pi}{L}x) + b_n \sin(\frac{n\pi}{L}x)\big)$ converges as follows for $N \to \infty$:*

$$s_N(x) \to \begin{cases} f(x), & \text{when } f \text{ is continuous at } x, \\ \frac{1}{2}(f(x+) + f(x-)), & \text{when } f \text{ has a jump at } x. \end{cases} \tag{1.3}$$

A better convergence property holds when f is more regular:

Theorem 1.2. *Let $f : \mathbb{R} \to \mathbb{R}$ have period $2L$ and be PC1C. Then the Fourier coefficients $a_n(f), b_n(f)$ of f, and the Fourier coefficients $a_n(f'), b_n(f')$ of f', are related by:*

$$\begin{aligned} a_n(f') &= \frac{n\pi}{L} b_n(f) \text{ for } n \geq 1, \quad a_0(f') = 0, \\ b_n(f') &= -\frac{n\pi}{L} a_n(f) \text{ for } n \geq 1. \end{aligned} \tag{1.4}$$

Moreover,

$$\sum_{n=1}^{\infty} (|a_n| + |b_n|) < \infty, \tag{1.5}$$

and the Fourier series of f converges uniformly (and absolutely) to f.

Proof. (This is taken from the proof of Theorem 2.9.3.) In this case, we see by integration by parts:

$$\begin{aligned} a_n(f') &= \frac{1}{L}\int_{-L}^{L} f'(x)\cos(\frac{n\pi}{L}x)\,dx = \frac{1}{L}\sum_{i=0}^{m}\int_{t_i}^{t_{i+1}} f'(x)\cos(\frac{n\pi}{L}x)\,dx \\ &= \frac{1}{L}\sum_{i=0}^{m}\Big(\int_{t_i}^{t_{i+1}} f(x)\frac{n\pi}{L}\sin(\frac{n\pi}{L}x)\,dx + \Big[f(x)\cos(\frac{n\pi}{L}x)\Big]_{t_i}^{t_{i+1}}\Big) \\ &= \frac{1}{L}\int_{-L}^{L} f(x)\frac{n\pi}{L}\sin(\frac{n\pi}{L}x)\,dx = \frac{n\pi}{L}b_n(f). \end{aligned} \tag{1.5}$$

We used in the calculation that the values of $f(x)\cos(\frac{n\pi}{L}x)$ at the points t_i (where f' may have jumps) cancel out in the summation, since $f(x)\cos(\frac{n\pi}{L}x)$ is continuous and has period $2L$. The proof that $b_n(f') = -\frac{n\pi}{L}a_n(f)$ is similar.

Bessel's inequality (Section 2.5) holds for f', so

$$\sum_{n=1}^{\infty} (|a_n(f')|^2 + |b_n(f')|^2) < \infty. \tag{1.6}$$

Using (1.4) and the inequality (for real or complex numbers)

$$|AB| \leq \tfrac{1}{2}(|A|^2 + |B|^2), \tag{1.7}$$

which holds since $(|A| - |B|)^2 \geq 0$, we find that for all N,

$$\begin{aligned} \sum_{n=1}^{N} (|a_n(f)| + |b_n(f)|) &= \frac{L}{\pi}\sum_{n=1}^{N}\frac{1}{n}(|b_n(f')| + |a_n(f')|) \\ &\leq \frac{L}{2\pi}\sum_{n=1}^{N}(\frac{1}{n^2} + |b_n(f')|^2) + \frac{L}{2\pi}\sum_{n=1}^{N}(\frac{1}{n^2} + |a_n(f')|^2) \\ &\leq \frac{L}{\pi}\sum_{n=1}^{\infty}\frac{1}{n^2} + \frac{L}{2\pi}\sum_{n=1}^{\infty}(|a_n(f')|^2 + |b_n(f')|^2) < \infty, \end{aligned} \tag{1.8}$$

(convergent series); this implies (1.5).

Then by Weierstrass' M-test, the Fourier series converges uniformly (and also absolutely, which means that the convergence holds for the series with the terms replaced by their absolute values). □

The information in (1.5) is useful more generally in discussions of convergence, when the separation of variables method is used.

Consider the boxed statement on page 119 in [A04] on the series solution to the wave equation with initial- and boundary conditions (1), (2), (3). For its full validity, several aspects must be discussed, such as:

(a) Under what hypotheses does the series (8) converge to a function $u(x,t)$?
(b) Under what hypotheses can $u(x,t)$ be checked to solve the wave equation (1)?
(c) Is the solution of (1), (2), (3) unique?

We shall not worry here about the uniqueness question, since it is more easily answered (positively) by other methods. But the first two questions are certainly important.

Take first the case where $g = 0$, so that all b_n^* are zero, and we are considering the expansion

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin(\frac{n\pi}{L}x)\cos(\frac{n\pi c}{L}t). \tag{1.9}$$

Since $|\sin(\frac{n\pi}{L}x)\cos(\frac{n\pi c}{L}t)| \le 1$ for all (x,t), the series (1.9) converges uniformly in (x,t) if

$$\sum_{n=1}^{\infty} |b_n| < \infty. \tag{1.10}$$

Here the b_n are the coefficients in the sinus-expansion of $f(x)$. Recall from Section 2.4 how it is constructed: We take the odd extension of f to the interval $[-L, L]$, and extend this to a function on $\mathbb{R}$ with period $2L$; let us denote the resulting function f^*. Then the sinus coefficients of f are precisely the Fourier coefficients of f^* (whose cosine terms vanish).

Now we can use Theorem 1.2 if f^* is PC1C. This holds precisely when f is PC1C on $[0, L]$ *with*

$$f(0) = 0, \quad f(L) = 0. \tag{1.11}$$

In that case, Theorem 1.2 assures that (1.10) holds, so (1.9) converges uniformly (in (x,t)) to a continuous function $u(x,t)$. Moreover, the value of u for $t = 0$ is $f(x)$ when $x \in [0, L]$.

When we want to check the initial condition $u_t(x, 0) = 0$, or the wave equation for u, we meet new difficulties: Can we perform differentiations by moving them past the summation sign (so that we can use that the individual terms do satisfy the equations)?

The most demanding question here is whether we can apply second derivatives in x and t. Recall from the theory of sequences of functions that when $h_n(x)$ is a sequence of C^1-functions such that $h_n(x) \to h(x)$ and $h'_n(x) \to k'(x)$ *uniformly* for $n \to \infty$, then $h(x)$ is C^1 with $h'(x) = k(x)$. For a series of C^1 functions this means that if *both the series and its termwise differentiated series* converge uniformly, then the sum of the termwise differentiated series equals the derivative of the function. (In other words: then "termwise differentiation is allowed".)

Termwise application of differentiations to (1.9) gives the series:

$$
\begin{aligned}
\frac{\partial}{\partial t}: &\quad -\sum_{n=1}^{\infty} b_n \frac{n\pi c}{L} \sin(\frac{n\pi}{L}x)\sin(\frac{n\pi c}{L}t),\\
\frac{\partial^2}{\partial t^2}: &\quad -\sum_{n=1}^{\infty} b_n (\frac{n\pi c}{L})^2 \sin(\frac{n\pi}{L}x)\cos(\frac{n\pi c}{L}t),\\
\frac{\partial}{\partial x}: &\quad \sum_{n=1}^{\infty} b_n \frac{n\pi}{L} \cos(\frac{n\pi}{L}x)\cos(\frac{n\pi c}{L}t),\\
\frac{\partial^2}{\partial x^2}: &\quad -\sum_{n=1}^{\infty} b_n (\frac{n\pi}{L})^2 \sin(\frac{n\pi}{L}x)\cos(\frac{n\pi c}{L}t).
\end{aligned}
\tag{1.12}
$$

These series will all converge uniformly if

$$
\sum_{n=1}^{\infty} n^2 |b_n| < \infty. \tag{1.13}
$$

Then the wave equation can be verified to hold. We note that the mixed (x,t)-derivative gives

$$
\frac{\partial^2}{\partial x \partial t}: \quad \sum_{n=1}^{\infty} b_n c(\frac{n\pi}{L})^2 \cos(\frac{n\pi}{L}x)\sin(\frac{n\pi c}{L}t),
$$

which also converges uniformly when (1.13) holds, so in fact $u(x,t)$ is then a C^2-function.

So we just have to look for criteria on f that assure (1.13). Here (1.4) comes in useful again. It shows that if f^* is PC1C, then $a_n(f^{*\prime}) = \frac{n\pi}{L} b_n(f^*)$, and if furthermore $f^{*\prime}$ is PC1C, then

$$
b_n(f^{*\prime\prime}) = -\frac{n\pi}{L} a_n(f^{*\prime}) = -(\frac{n\pi}{L})^2 b_n(f^*). \tag{1.14}
$$

Thus

$$
\sum_{n=1}^{N} n^2 |b_n(f^*)| = \frac{L^2}{\pi^2} \sum_{n=1}^{N} |b_n(f^{*\prime\prime})|, \tag{1.15}
$$

so (1.13) is assured if (1.10) holds for $f^{*\prime\prime}$. This is true when $f^{*\prime\prime}$ is PC1C, by Theorem 1.2.

To sum up, we get (1.13) when f^*, $f^{*\prime}$ and $f^{*\prime\prime}$ are PC1C (in other words when f^* is C^2 with a piecewise continuous $f^{*\prime\prime\prime}$). For f itself this holds when f, f' and f'' are PC1C on $[0,L]$ with

$$
f(0) = f(L) = f''(0) = f''(L) = 0. \tag{1.16}
$$

There is no extra condition on f' at the endpoints, since $f^{*\prime}$ is even (as the derivative of an odd function), hence continuous across 0 and L.

We must also consider the case where $f = 0$ and g is nonzero in the problem (1), (2), (3). (The solution for general f and g is found as the sum of the solutions for the cases with $\{f, 0\}$ and $\{0, g\}$.) Here

$$u(x,t) = \sum_{n=1}^{\infty} b_n^* \sin(\frac{n\pi}{L}x)\sin(\frac{n\pi c}{L}t), \tag{1.17}$$

and termwise applications of differentiations give

$$\begin{aligned}
\frac{\partial}{\partial t} &: \quad \sum_{n=1}^{\infty} b_n^* \frac{n\pi c}{L} \sin(\frac{n\pi}{L}x)\cos(\frac{n\pi c}{L}t), \\
\frac{\partial^2}{\partial t^2} &: \quad -\sum_{n=1}^{\infty} b_n^* \big(\frac{n\pi c}{L}\big)^2 \sin(\frac{n\pi}{L}x)\sin(\frac{n\pi c}{L}t), \\
\frac{\partial}{\partial x} &: \quad \sum_{n=1}^{\infty} b_n^* \frac{n\pi}{L} \cos(\frac{n\pi}{L}x)\sin(\frac{n\pi c}{L}t), \\
\frac{\partial^2}{\partial x^2} &: \quad -\sum_{n=1}^{\infty} b_n^* \big(\frac{n\pi}{L}\big)^2 \sin(\frac{n\pi}{L}x)\sin(\frac{n\pi c}{L}t) \\
\frac{\partial^2}{\partial x \partial t} &: \quad \sum_{n=1}^{\infty} b_n^* c \big(\frac{n\pi}{L}\big)^2 \cos(\frac{n\pi}{L}x)\cos(\frac{n\pi c}{L}t).
\end{aligned} \tag{1.18}$$

These series will all converge uniformly if

$$\sum_{n=1}^{\infty} n^2 |b_n^*| < \infty. \tag{1.19}$$

Note here that the n'th sinus coefficient of g is $\lambda_n b_n^* = \frac{n\pi c}{L} b_n^*$. So when g^* is PC1C, we already have that $\sum_{n=1}^{\infty} n|b_n^*| < \infty$, by Theorem 1.2. We just need also $g^{*\prime}$ to be PC1C to get (1.19). The conditions on g are then: g and g' are PC1C on $[0, L]$ with

$$g(0) = g(L) = 0. \tag{1.20}$$

The findings are summed up in a theorem:

Theorem 1.3. *When f, f', f'' are PC1C on $[0, L]$ satisfying (1.16), and g, g' are PC1C on $[0, L]$ satisfying (1.20), then $u(x, t)$ in (8) is C^2 and solves (1), (2), (3).*

Such a solution is called a “classical solution”; it verifies the given differential equation and boundary conditions, applied in the original sense of taking derivatives.

Note that the book [A04] goes on to discuss some examples, which *do not* satisfy the conditions in Theorem 1.3! The function f in Example 1 of Section 3.3 is only PC1C, its first derivative has a jump at $x = \frac{1}{3}$ and its second derivative does not have a meaning there. Also when d'Alembert's method is applied to this f (Example 2 in Section 3.4), there is something that is not in order, since it is not two times differentiable (so the calculations on page 127 are not justified at all points).

There exists a deeper and more refined theory of differentiation called Distribution Theory, where the operations can be given a rigorous meaning. It is far beyond our means at this level (but the interested reader can watch out for later possibilities to learn about it). The book [A04] has some introductory remarks on it in Section 7.8.

For the moment, we can use the words "generalized solution" to describe functions that come out of our methods but cannot be completely verified to satisfy the desired equations and conditions.

The solution formulas of d'Alembert (Section 3.4) make it possible to discuss some generalized solutions in a more satisfactory way.

Again, when we use the formulas for a given pair of initial values f and g, we must assume some smoothness of the odd periodic extensions f^* and g^* in order to verify that the differential equation is satisfied, see p. 127. The calculations there are only fully meaningful when f^* is two times differentiable and g^* is differentiable.

But d'Alembert's method has the advantage that we can see much more clearly what goes wrong when $f^{*\prime\prime}$ or $g^{*\prime}$ does not exist.

Consider again the problem with the triangular function f (and $g = 0$) in Examples 3.3.1 and 3.4.2. The odd 2-periodic extension f^* is linear, hence C^∞, *except* at the points $2k \pm \frac{1}{3}$, $k \in \mathbb{Z}$, stemming from one point $\frac{1}{3}$ in the original interval $[0, 1]$. We can follow the effect in the solution

$$u(x,t) = \tfrac{1}{2}(f^*(x+ct) + f^*(x-ct)), \tag{1.21}$$

by thinking of which points (x, t) are *influenced by* the point $x = \frac{1}{3}$ in $[0, 1]$ on the x-axis. In the first region I (see Figure 4 of Section 3.4), these lie on the two characteristic lines emanating from $(\frac{1}{3}, 0)$. Higher up in the figure, we reflect these characteristic line in the vertical boundaries (this corresponds to fetching the information from the point on the x-axis found by reflecting $(\frac{1}{3}, 0)$ around 0 or 1); still higher up we have reflected the characteristic lines several times (which corresponds to fetching the information from points on the x-axis of the form $2k \pm \frac{1}{3}$ with more general k).

The fundamental observation is that outside these reflected characteristic lines, $u(x, t)$ is fine! It can be differentiated as much as we want, and satisfies the wave equation.

So the solution is only singular in a small set compared to where it is regular. At any given level $t = t_0$, there are at most two points in $[0, 1]$ where differentiation fails.

This demonstrates an important property of the wave equation, the "propagation of singularities" property: Singularities in the initial data are propagated only along the characteristics. This is found also in higher dimensional wave equations, for example when x runs in $\mathbb{R}^3$. Let us just mention, without further explanation, that there is then a cone in (x, t)-space attached to each $(x, 0)$ (the "future cone" $\{(x, t) \mid |x| = ct\}$) such that a singularity at x for $t = 0$, at a later time $t = t_0$ gives singularities at the intersection of the cone with the hyperplane $t = t_0$ (and not elsewhere).

1.2 Convergence questions for the one-dimensional heat equation.

For the solution to the one-dimensional heat equation with initial- and boundary conditions listed on page 138 in [A04], we can ask exactly the same convergence questions as we did for the wave equation in the preceding section.

Since $|\sin\frac{n\pi}{L}x)| \le 1$ for all x and $|e^{-\lambda_n^2 t}| \le 1$ for $t \ge 0$, the series (4) describing u,

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin(\frac{n\pi}{L}x)e^{-\lambda_n^2 t}, \tag{1.22}$$

converges uniformly in $(x,t) \in [0,L] \times [0,\infty[$ when the odd periodic extension f^* is PC1C, i.e. when f is PC1C on $[0,L]$ satisfying (1.11). The initial- and boundary conditions are then likewise satisfied. For the verification of the differential equation, we calculate the termwise differentiated series:

$$\begin{aligned} \frac{\partial}{\partial t} &: \quad -\sum_{n=1}^{\infty} b_n \lambda_n^2 \sin(\frac{n\pi}{L}x)e^{-\lambda_n^2 t}, \\ \frac{\partial}{\partial x} &: \quad \sum_{n=1}^{\infty} b_n \frac{n\pi}{L} \cos(\frac{n\pi}{L}x)e^{-\lambda_n^2 t}, \\ \frac{\partial^2}{\partial x^2} &: \quad -\sum_{n=1}^{\infty} b_n (\frac{n\pi}{L})^2 \sin(\frac{n\pi}{L}x)e^{-\lambda_n^2 t}, \end{aligned} \tag{1.23}$$

where we recall that $\lambda_n = \frac{n\pi c}{L}$. They converge uniformly when (1.13) holds, so we see that all conditions are verified by $u(x,t)$ when f is as in Theorem 1.3.

However, the solutions of the heat equation have much more smoothness than the solutions of the wave equation, as we shall now see. (This is to some extent demonstrated in Example 3.5.1, but we shall prove the general result.)

Assume just that f is, say, piecewise continuous, so that the Fourier series satisfies the Parseval identity (converges in the mean to f). The coefficients must then necessarily satisfy

$$a_n \to 0, \quad b_n \to 0, \quad \text{for } n \to \infty, \tag{1.24}$$

and in particular there is a constant C so that

$$|a_n| \le C, \quad |b_n| \le C, \quad \text{for all } n. \tag{1.25}$$

Now consider $u(x,t)$ for $x \in [0,L]$ and $t \ge \varepsilon$, for some $\varepsilon > 0$. Since

$$e^{-\lambda_n^2 t} \le e^{-\lambda_n^2 \varepsilon} = e^{-C'n^2} \text{ with } C' = \frac{\pi^2 c^2 \varepsilon}{L^2} > 0, \tag{1.26}$$

the series in (1.23) will converge uniformly on $D_\varepsilon = \{(x,t) \mid x \in [0,L],\, t \ge \varepsilon\}$ if

$$\sum_{n=1}^{\infty} n^2 C e^{-C'n^2} < \infty, \tag{1.27}$$

and this holds since $e^{-C'n^2}$ goes very fast to zero; in fact, for any k,

$$n^k e^{-C'n^2} \le C_k n^{-2}, \tag{1.28}$$

for a suitable C_k. So u verifies the heat equation on D_ε for any $\varepsilon > 0$, under just the assumption that the sinus coefficients of f are bounded.

We can carry this still further:

Take any termwise (x,t)-derivative of the series for u:

$$\frac{\partial^j}{\partial t^j}\frac{\partial^k}{\partial x^k}: \quad \pm\sum_{n=1}^{\infty} b_n(-\lambda_n^2)^j\left(\frac{n\pi}{L}\right)^k \begin{matrix}\sin\\ \cos\end{matrix}\left(\frac{n\pi}{L}x\right)e^{-\lambda_n^2 t}; \tag{1.29}$$

it converges uniformly on D_ε, since

$$\sum_{n=1}^{\infty} n^{2j+k}e^{-C'n^2} < \infty. \tag{1.30}$$

So in fact, $u(x,t)$ is C^∞ for $t > 0$, under quite weak assumptions on f !

Also the boundary conditions $u(0,t) = 0$, $u(L,t) = 0$, are verified for $t > 0$.

We have shown:

Theorem 1.4. *The function $u(x,t)$ in (1.22) is C^∞ for $(x,t) \in [0,L]\times\,]0,\infty[$ and satisfies the heat quation and the boundary conditions there, when f is merely piecewise continuous.*

Observe that there is a radical difference from the situation of the wave equation: For the wave equation, singularities in the initial data were kept alive when t was increasing (however in a controlled way). For the heat equation, singularities in the initial value are killed immediately, as t becomes larger than 0.

Another difference is also worth pointing out: The series in (1.22) will very rarely be convergent when $t < 0$, for then the powers in the exponential functions are positive and the values grow very fast for $n \to \infty$. Briefly speaking: The "backwards heat equation" does not have good solvability properties! (If one does not worry about convergence, one might not get this point.)

In contrast with this, we could easily take $t < 0$ in the wave equation; both the Fourier series solution and d'Alembert's solution can be discussed in the same way for $t < 0$ as for $t > 0$. So the backwards wave equation has similar solvability properties as the forwards wave equation.

1.3 Convergence questions for the Laplace equation on a rectangle.

For the Laplace equation on a rectangle $D = [0,a]\times[0,b]$ one poses a boundary condition on all four sides. The variable t is called y here since it enters on equal terms with x. Note that in contrast with heat and wave equations we pose a condition both at $y = 0$ and at $y = b$. There is no extra factor c^2.

The basic case to treat is where a function $f_2(x)$ is prescribed at $y = b$ and the solution is required to be 0 on the other edges. The resulting formula is (2) on page 165:

$$u(x,y) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{a}x\right)\sinh\left(\frac{n\pi}{a}y\right), \tag{1.31}$$

where the B_n are determined by

$$\begin{aligned} B_n &= \frac{1}{\sinh(\frac{n\pi}{a}b)}\,\frac{2}{a}\int_0^a f(x)\sin\left(\frac{n\pi}{a}x\right)dx \\ &= \frac{1}{\sinh(\frac{n\pi}{a}b)}\,b_n; \end{aligned} \tag{1.32}$$

b_n being the n'th sinus coefficient of f_2.

For $y = b$,

$$u(x,b) = \sum_{n=1}^{\infty} b_n \sin(\frac{n\pi}{a}x) = f_2(x), \tag{1.33}$$

converging as in Theorems 1.1 or 1.2 when f_2^* (the odd $2a$-periodic extension of f_2) satisfies the hypoteses there. We find moreover that if f_2 is as f in Theorem 1.3 (with $L = a$), the series (1.31) converges in C^2 on D.

For $y < b$, we can do *better*, much like for the heat equation: Let $y \in [0, b-\varepsilon]$ for some $\varepsilon > 0$. Then we have the inequalities

$$\begin{aligned} 0 \le \frac{\sinh(\frac{n\pi}{a}y)}{\sinh(\frac{n\pi}{a}b)} &= \frac{e^{\frac{n\pi}{a}y} - e^{-\frac{n\pi}{a}y}}{e^{\frac{n\pi}{a}b} - e^{-\frac{n\pi}{a}b}} = e^{\frac{n\pi}{a}(y-b)} \frac{1 - e^{-2\frac{n\pi}{a}y}}{1 - e^{-2\frac{n\pi}{a}b}} \\ &\le e^{-\frac{n\pi}{a}\varepsilon} \frac{1}{1 - e^{-2\frac{\pi}{a}b}}\,, \\ 0 \le \frac{\cosh(\frac{n\pi}{a}y)}{\sinh(\frac{n\pi}{a}b)} &= \frac{e^{\frac{n\pi}{a}y} + e^{-\frac{n\pi}{a}y}}{e^{\frac{n\pi}{a}b} - e^{-\frac{n\pi}{a}b}} \le e^{-\frac{n\pi}{a}\varepsilon} \frac{2}{1 - e^{-2\frac{\pi}{a}b}}\,. \end{aligned} \tag{1.34}$$

So, with $C' = \frac{\pi\varepsilon}{a}$ and $C'' = \frac{2}{1-e^{-2\pi b/a}}$, we have that

$$\frac{\sinh(\frac{n\pi}{a}y)}{\sinh(\frac{n\pi}{a}b)} \text{ and } \frac{\cosh(\frac{n\pi}{a}y)}{\sinh(\frac{n\pi}{a}b)} \le C'' e^{-C'n} \text{ for all } n, \text{ when } y \in [0, b-\varepsilon]. \tag{1.35}$$

Then on $D_\varepsilon = [0,a] \times [0, b-\varepsilon]$, the series for u has the majorizing series

$$\sum_{n=1}^{\infty} |b_n| C'' e^{-C'n},$$

which converges when merely the sequence of $|b_n|$ is bounded. Application of derivatives $\frac{\partial^j}{\partial x^j}\frac{\partial^k}{\partial y^k}$ to (1.31) gives series

$$\pm \sum (\frac{n\pi}{a})^{j+k} \frac{b_n}{\sinh(\frac{n\pi}{a}b)} \begin{matrix}\sin\\ \cos\end{matrix} (\frac{n\pi}{a}x) \begin{matrix}\sinh\\ \cosh\end{matrix} (\frac{n\pi}{a}y) \tag{1.36}$$

where cos resp. cosh is used when j resp. k is odd. In view of (1.35) they have majorizing series

$$\text{const.} \sum_{n=1}^{\infty} n^{j+k} |b_n| e^{-C'n}, \tag{1.37}$$

for $(x,y) \in D_\varepsilon$, which are all convergent when $|b_n|$ is bounded. (This holds e.g. when f_2 is piecewise continuous.)

We conclude that the Fourier series solution converges, with all its derivatives, to a C^∞ function for $y \le b - \varepsilon$, any ε, under very mild hypotheses on f_2.

For the solution on page 168, where nonzero functions are given on all four edges, we conclude that there is uniform convergence of all derivatives on compact subsets of the interior $D^\circ =]0,a[\times]0,b[$. We have hereby obtained:

Theorem 1.5. *The series in* [A04, (9) p. 168] *converges to a* C^∞*-function* $u(x,y)$ *on* $]0,a[\times]0,b[$ *and satisfies the Laplace equation there, when* f_1, f_2, g_1 *and* g_2 *are merely piecewise continuous (in fact, all the derived series converge uniformly on compact subsets of* $]0,a[\times]0,b[$*).*

The series converges uniformly on $[0,a]\times[0,b]$ *to a continuous function satisfying the boundary conditions, when* f_1, f_2, g_1 *and* g_2 *are PC1C and are zero at the endpoints.*

So when all the conditions in the theorem are satisfied, u is a classical solution (it is customary to require verification of the differential equation only in the interior of the domain). Uniform convergence on D can also be obtained when the functions on the boundary are allowed to have other values in the corners, as long as they match to define a continuous function on the boundary, since this case can be reduced to the case where the corner values are zero by subtraction of a suitable function of the form $c_1+c_2x+c_3y+c_4xy$.

When the functions on the boundary are piecewise C^1 and take the mean values at the jumps, one can also view the solution as a classical solution (however with different qualities of the convergence at the various points). When the functions on the boundary are merely piecewise continuous, their value is assumed in a square mean sense on each edge; this is better viewed as a generalized solution.

E1. Exercises

Exercise E1.1. Consider the function

$$h_k(x) = x^k(1-x)^k, \quad x\in[0,1],$$

where k is a positive integer.

For the boundary value problem for the wave equation [A04, 3.3.(1)–(3)] with $L=1$, $c=1$ and initial data $f=h_k$, $g=0$, find out for which values of k the solution is seen to be classical, when it is constructed by:

(1) the Fourier series method (separation of variables)?
(2) d'Alembert's method?

Explain your argumentation.

Find the value of the solution at the point $(x,t)=(\frac{1}{2},1)$.

Exercise E1.2. Same questions as in Exercise E1.1, now with initial data $f=0$, $g=h_k$.

Exercise E1.3. Find the series solution of the boundary value problem for the heat equation [A04, 3.5.(1)–(3)] with $L=1$, $f(x)=x(1-x)$, and discuss its convergence.

Exercise E1.4. Show that any function $c_1+c_2x+c_3y+c_4xy$ satisfies the Laplace equation on $\mathbb{R}^2$.

Exercise E1.5. Show that the Dirichlet problem in Figure 1 on page 164 in [A04] has a classical solution, when

$$a=b=1,\quad f_1(x)=x^2,\ f_2(x)=1,\ g_1(y)=y^2,\ g_2(y)=1,$$

and find the solution.

2. Fourier expansions in higher dimensions

2.1 Multiple Fourier series.

The theory of Fourier expansions extends readily to higher dimensions. Here the complex formulation is advantageous, because it gives simpler formulas (allowing a better overview than when multiple products of cosines and sines occur everywhere).

Before presenting this, let us underline the fact that is put forward in [A04, Section 2.5], that any $2p$-periodic function f that is *square integrable* on the interval $[-p, p]$ can be expanded in a Fourier series, with coefficients determined by the Euler formulas on page 39. Moreover, the Bessel inequality and Parseval identity hold for f. Special cases are piecewise continuous functions, or just bounded (measurable) functions. It can be seen directly from the Euler formulas that the Fourier coefficients are bounded in n, but the Bessel inequality gives a still better information, namely that $a_n \to 0$ and $b_n \to 0$ for $n \to \infty$.

Now recall the complex formulation in one variable: It is based on the Euler identity, for $x \in \mathbb{R}$:

$$\begin{aligned} e^{ix} &= \cos x + i \sin x, \quad \text{hence} \\ \cos x = \tfrac{1}{2}(e^{ix} + e^{-ix}), &\quad \sin x = \tfrac{1}{2i}(e^{ix} - e^{-ix}). \end{aligned} \tag{2.1}$$

In the Fourier series of a $2p$-periodic function $f(x)$,

$$f(x) = a_0 + \sum_{n=1}^{\infty} \big(a_n \cos(\tfrac{n\pi}{p} x) + b_n \sin(\tfrac{n\pi}{p} x)\big), \tag{2.2}$$

we can insert the replacements

$$\begin{aligned} \cos(\tfrac{n\pi}{p} x) &= \tfrac{1}{2}(e^{i\frac{n\pi}{p}x} + e^{-i\frac{n\pi}{p}x}), \\ \sin(\tfrac{n\pi}{p} x) &= \tfrac{1}{2i}(e^{i\frac{n\pi}{p}x} - e^{-i\frac{n\pi}{p}x}); \end{aligned} \tag{2.3}$$

then

$$s_N(x) = a_0 + \sum_{n=1}^{N} \big(a_n \cos(\tfrac{n\pi}{p} x) + b_n \sin(\tfrac{n\pi}{p} x)\big) = \sum_{m=-N}^{N} c_m e^{i\frac{m\pi}{p}x}, \tag{2.4}$$

with

$$c_0 = a_0, \quad c_n = \tfrac{1}{2}(a_n - ib_n), \quad c_{-n} = \tfrac{1}{2}(a_n + ib_n). \tag{2.5}$$

This justifies writing (2.2) as

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{i\frac{n\pi}{p}x}. \tag{2.6}$$

One has that

$$c_n = \frac{1}{2p} \int_{-p}^{p} f(x) e^{-i\frac{n\pi}{p}x}\, dx \text{ for all } n \in \mathbb{Z}, \tag{2.7}$$

which holds also when complex-valued functions $f(x)$ are allowed (still with $x \in \mathbb{R}$). The Parseval identity is:

$$\sum_{n=-\infty}^{\infty} |c_n|^2 = \frac{1}{2p} \int_{-p}^{p} |f(x)|^2 \, dx. \tag{2.8}$$

Theorem 1.2 says in the complex formulation that when f is PC1C with period $2p$, then:

$$\begin{aligned} &\text{(i)} \quad c_n(f') = i\frac{n\pi}{p} c_n(f) \text{ for all } n \in \mathbb{Z}, \\ &\text{(ii)} \quad \sum_{n=-\infty}^{\infty} |c_n| < \infty, \end{aligned} \tag{2.9}$$

and (iii) the Fourier series converges uniformly (and absolutely) to f.

It is not hard to extend the ideas to higher dimensions. For simplicity in the formulas we now let $p = \pi$ and leave to the reader to do the scaling when other lengths are needed.

On $\mathbb{R}^k$ with points denoted $x = (x_1, \dots, x_k)$ we consider functions $f(x)$ that have period 2π in each variable $x_1, \dots, x_k$. They are completely determined by their values on the cube $[-\pi, \pi]^k$. The elements of $\mathbb{Z}^k$ will be denoted $\mathbf{n} = (n_1, \dots, n_k)$, with length

$$\|\mathbf{n}\| = \sqrt{n_1^2 + \cdots + n_k^2}\,. \tag{2.10}$$

The functions

$$e^{i\mathbf{n}\cdot x} = e^{i(n_1 x_1 + \dots n_k x_k)}, \quad \mathbf{n} \in \mathbb{Z}^k, \tag{2.11}$$

are 2π-periodic in each variable x_j and satisfy

$$(e^{i\mathbf{n}\cdot x}, e^{i\mathbf{m}\cdot x}) = \begin{cases} 0 \text{ if } \mathbf{n} \neq \mathbf{m}, \\ (2\pi)^k \text{ if } \mathbf{n} = \mathbf{m}, \end{cases} \tag{2.12}$$

when we use the scalar product (inner product)

$$\begin{aligned} (f,g) &= \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} f(x_1, x_2, \dots, x_k) \overline{g}(x_1, x_2, \dots, x_k) \, dx_1 dx_2 \dots dx_k \\ &= \int_{[-\pi,\pi]^k} f(x) \overline{g}(x) \, dx. \end{aligned}$$

Indeed,

$$(e^{i\mathbf{n}\cdot x}, e^{i\mathbf{m}\cdot x}) = \int_{-\pi}^{\pi} e^{in_1 x_1} e^{-im_1 x_1} \, dx_1 \cdots \int_{-\pi}^{\pi} e^{in_k x_k} e^{-im_k x_k} \, dx_k;$$

here if $n_j \neq m_j$ for some j, the integral in x_j gives a factor 0; on the other hand if $n_j = m_j$ for all j, each integral over $[-\pi, \pi]$ contributes with a factor 2π.

It is shown on the basis of the one-dimensional result that a square integrable function $f(x)$ on $]-\pi,\pi]^k$, extended to be 2π-periodic in each variable $x_1,\dots,x_k$, can be expanded in a Fourier series

$$
\begin{aligned}
f(x) &\sim \sum_{\mathbf{n}\in\mathbb{Z}^k} c_{\mathbf{n}} e^{i\mathbf{n}\cdot x}, \text{ where} \\
c_{\mathbf{n}} &= \frac{1}{(2\pi)^k}\int_{[-\pi,\pi]^k} f(x)e^{-i\mathbf{n}\cdot x}\,dx,
\end{aligned}
\tag{2.13}
$$

in such a way that the partial sum

$$
s_N(x) = \sum_{\max\{|n_1|,\dots|n_k|\}\le N} c_{\mathbf{n}} e^{i\mathbf{n}\cdot x}
\tag{2.14}
$$

converges in the mean to $f(x)$, in the sense that

$$
\int_{[-\pi,\pi]^k} |f(x)-s_N(x)|^2\,dx \to 0 \text{ for } N\to\infty.
\tag{2.15}
$$

Here the following Parseval identity holds:

$$
\sum_{\mathbf{n}\in\mathbb{Z}^k} |c_{\mathbf{n}}|^2 = \frac{1}{(2\pi)^k}\int_{[-\pi,\pi]^k} |f(x)|^2\,dx.
\tag{2.16}
$$

As a corollary to the Parseval identity we see that $|c_{\mathbf{n}}|\to 0$ for $\|\mathbf{n}\|\to\infty$; this holds under the mere assumption that f is square integrable on $[-\pi,\pi]^k$. We give below some information on uniform convergence.

For $k=2$, the formulation with cosine and sine is found from the above by noting that

$$
e^{i(n_1x_1+n_2x_2)} = (\cos n_1x_1 + i\sin n_1x_1)(\cos n_2x_2 + i\sin n_2x_2).
$$

For n_1 and $n_2\in\mathbb{N}$ we can use this in the four terms

$$
\begin{aligned}
c_{(n_1,n_2)}e^{i(n_1x_1+n_2x_2)} &+ c_{(n_1,-n_2)}e^{i(n_1x_1-n_2x_2)} \\
&+ c_{(-n_1,n_2)}e^{i(-n_1x_1+n_2x_2)} + c_{(-n_1,-n_2)}e^{i(-n_1x_1-n_2x_2)}
\end{aligned}
$$

and regroup them as a linear combination of $\cos n_1x_1\cos n_2x_2$, $\cos n_1x_1\sin n_2x_2$, $\sin n_1x_1\cos n_2x_2$ and $\sin n_1x_1\sin n_2x_2$. This is somewhat unmanageable, but it becomes more manageable when we restrict the attention to functions that are *odd* in x_1 as well as x_2; they only have sine terms

$$
\begin{aligned}
f(x) &\sim \sum_{n_1,n_2\in\mathbb{N}} b_{n_1,n_2}\sin n_1x_1\sin n_2x_2, \text{ with} \\
b_{n_1,n_2} &= -c_{(n_1,n_2)} + c_{(n_1,-n_2)} + c_{(n_1,-n_2)} - c_{(-n_1,-n_2)},
\end{aligned}
\tag{2.17}
$$

since, in the calculation of (2.13), a cosine factor $\cos n_ix_i$ integrated together with f in the x_i-variable gives 0. Here

$$
b_{n_1,n_2} = \frac{4}{\pi^2}\int_0^\pi\int_0^\pi f(x_1,x_2)\sin n_1x_1\sin n_2x_2\,dx_1dx_2.
$$

Note however that when one differentiates a sine series, cosine comes in again.

There is a general result:

Theorem 2.1.

1° *If $f(x)$ is 2π-periodic in each coordinate, and C^1, then for all $\mathbf{n} \in \mathbb{Z}^k$,*

$$c_{\mathbf{n}}\big(\frac{\partial f}{\partial x_j}\big) = in_j c_{\mathbf{n}}(f), \quad j = 1, \dots, k, \tag{2.18}$$

and the Parseval identity for the derivatives implies

$$\sum_{\mathbf{n}\in\mathbb{Z}^k} \|\mathbf{n}\|^2 |c_{\mathbf{n}}(f)|^2 < \infty. \tag{2.19}$$

Moreover, if f is C^l for some $l \geq 1$, then

$$\sum_{\mathbf{n}\in\mathbb{Z}^k} \|\mathbf{n}\|^{2l} |c_{\mathbf{n}}(f)|^2 < \infty. \tag{2.20}$$

2° *For $k = 2$ or 3, if $f(x)$ is 2π-periodic in each coordinate and C^{l+2}, then*

$$\sum_{\mathbf{n}\in\mathbb{Z}^k} \|\mathbf{n}\|^{l} |c_{\mathbf{n}}(f)| < \infty. \tag{2.21}$$

The estimate (2.21) *implies that the Fourier series and it termwise differentiated series up to order l are uniformly convergent.*

Indications of proof. In 1°, the identity in (2.18) is shown by integration by parts (in the x_j-variable) in the formula for $c_{\mathbf{n}}(\frac{\partial f}{\partial x_j})$. Then the Parseval identity for $\frac{\partial f}{\partial x_j}$ implies the convergence of the series $\sum_{\mathbf{n}} |n_j|^2 |c_{\mathbf{n}}(f)|^2$. When we sum over j we find (2.19). When f is C^l, this can be applied for any succession of l partial derivatives, showing that

$$\sum_{\mathbf{n}\in\mathbb{Z}^k} |p(n_1, \dots, n_k)|^2 |c_{\mathbf{n}}(f)|^2 < \infty$$

for any polynomial p of degree l. Since $\|\mathbf{n}\|^{2l}$ is bounded by a sum of squares of such polynomials, the result (2.20) follows.

For 2°, note that it is here a question of series with $|c_{\mathbf{n}}|$ in the first power only. One can show that the series $\sum_{\mathbf{n}\in\mathbb{Z}^k\setminus\{\mathbf{0}\}} \|\mathbf{n}\|^{-4}$ is convergent for $k = 2$ and 3 (this is related to the fact that $\int_{|x|\geq 1} |x|^{-4}\,dx < \infty$ in dimensions < 4; see also Exercise E2.3). Then we can do a trick as in the proof of Theorem 1.2:

$$\begin{aligned}\sum_{\|\mathbf{n}\|\leq N} \|\mathbf{n}\|^{l} |c_{\mathbf{n}}| &= \sum_{0<\|\mathbf{n}\|\leq N} \|\mathbf{n}\|^{-2} \|\mathbf{n}\|^{l+2} |c_{\mathbf{n}}| \\ &\leq \tfrac{1}{2}\Big(\sum_{0<\|\mathbf{n}\|\leq N} \|\mathbf{n}\|^{-4} + \sum_{0<\|\mathbf{n}\|\leq N} \|\mathbf{n}\|^{2(l+2)} |c_{\mathbf{n}}|^2 \Big) \leq C,\end{aligned}$$

for all N, where we for the last series use that (2.20) holds with l replaced by $l + 2$.

The estimate (2.21) implies uniform convergence of the termwise differentiated series up to order l, since $|e^{i\mathbf{n}\cdot x}| = 1$, and

$$\frac{\partial}{\partial x_j} e^{i\mathbf{n}\cdot x} = in_j e^{i\mathbf{n}\cdot x}, \tag{2.22}$$

so (2.21) (times a constant) is a majorizing series for all those termwise derived series. □

Remark 2.2.

(a) There is a result along the lines of 2° also in higher dimensions. One can show that for general k, the series $\sum_{\mathbf{n}\in\mathbb{Z}^k\setminus\{\mathbf{0}\}} \|\mathbf{n}\|^{-2k'}$ is convergent when $2k' > k$. Then when the Fourier coefficients of f satisfy the estimate (2.20) with l replaced by $l + k'$, they will satisfy (2.21), by a version of the above trick.

This is actually an example of Sobolev's Theorem — prominent in the more advanced theory — that says that functions with finite Sobolev norm of order $l+k'$, some $k' > k/2$, are in C^l (here the squareroot of (2.20) plays the role of the l'th Sobolev norm).

(b) In part 1° of Theorem 2.1, the formulas (2.18)–(2.19) can be shown under slightly weaker assumptions. It suffices that f is continuous with square integrable first derivatives defined in some reasonable sense. For example, if $[-\pi, \pi]^k$ is divided into a finite number of polyedric subdomains, and the first derivatives of f are defined in each of these subdomains and extend to continuous functions on their closures, then (2.18) and (2.19) hold. We can call such derivatives *piecewise continuous* (although the notion could be defined also in more general situations). Similarly, if f is C^{l-1} and the l'th order derivatives are piecewise continuous, then (2.20) holds.

In Theorem 2.1 2°, it is then sufficient for (2.21) that f is C^{l+1} with piecewise continuous derivatives of order $l + 2$.

2.2 The wave equation with initial data on a rectangle.

Using Theorem 2.1, we can justify the solution formula for the two-dimensional wave equation in [A04, Section 3.7] as follows (using the notation (x, y) for a point in $\mathbb{R}^2$):

Theorem 2.3. 1° *When $f(x, y)$ is C^2 and $g(x, y)$ is C^1 on $M = [0, a] \times [0, b]$, and f and g are zero on the boundary $\partial M = \{\, (x, y) \in M \mid x = 0$ or a, $y = 0$ or $b \,\}$, then*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(|B_{mn}| + |B^*_{mn}|) < \infty. \tag{2.23}$$

Then the series in (4) *converges uniformly on $M \times [0, \infty[$ to a continuous function $u(x, y, t)$ satisfying the boundary condition $u = 0$ for $(x, y) \in \partial M$ and the first initial condition $u = f$ for $t = 0$.*

2° *When furthermore $f(x, y)$ is C^4 and $g(x, y)$ is C^3 on M, and*

$$\frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial y^2}, \frac{\partial^2 g}{\partial x^2} \text{ and } \frac{\partial^2 g}{\partial y^2} \text{ are } 0 \text{ on } \partial M, \tag{2.24}$$

then

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(m^2 + n^2)(|B_{mn}| + |B^*_{mn}|) < \infty, \tag{2.25}$$

and $u(x, y, t)$ is C^2 on $M \times [0, \infty[$ and satisfies the wave equation and the initial- and boundary conditions.

Proof. When f and g satisfy the hypotheses in 1°, they extend to functions f^* resp. g^* on $\mathbb{R}^2$ that are odd in x with period $2a$, and odd in y with period $2b$, such that the extended functions are in C^1 on $\mathbb{R}^2$ and the second derivatives of f^* are piecewise continuous. The

B_{mn} are the coefficients in the sine expansion of f in two variables as in (2.17), so by Theorem 2.1 2° with $l = 0$ and Remark 2.2(b), the series $\sum_{m,n} |B_{mn}|$ is convergent. The B^*_{mn} satisfy

$$B^*_{mn} = \frac{b^*_{mn}}{\lambda_{mn}}, \tag{2.26}$$

where the b^*_{mn} are the coefficients in the sine expansion of g in two variables. By 1° of Theorem 2.1,

$$\sum_{m,n\in\mathbb{N}} (m^2 + n^2)|b^*_{mn}|^2 < \infty. \tag{2.27}$$

Since $\lambda_{mn} = c\pi\sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}}$ clearly satisfies

$$c_1(m^2 + n^2)^{\frac{1}{2}} \le \lambda_{mn} \le c_2(m^2 + n^2)^{\frac{1}{2}} \tag{2.28}$$

with positive constants c_1 and c_2, we conclude from (2.26) and (2.27) that the series of $|B^*_{mn}|^2$ satisfies

$$\sum_{m,n\in\mathbb{N}} (m^2 + n^2)^2|B^*_{mn}|^2 < \infty. \tag{2.29}$$

From this we deduce as in the proof of Theorem 2.1 2° that

$$\sum_{m,n\in\mathbb{N}} |B^*_{mn}| < \infty. \tag{2.30}$$

This completes the proof of (2.23), which implies uniform convergence as stated.

When furthermore the hypotheses of 2° are satisfied, the extensions f^* and g^* are C^3 on $\mathbb{R}^2$ and the fourth derivatives of f^* are piecewise continuous. It follows from Theorem 2.1 2° and Remark 2.2(b) that

$$\sum_{m,n}(m^2 + n^2)|B_{mn}| < \infty, \quad \sum_{m,n}(m^2 + n^2)^{\frac{1}{2}}|b^*_{mn}| < \infty. \tag{2.30}$$

Using again (2.26) and (2.28) we conclude that (2.25) holds. Then the termwise differentiated series up to order 2 converge uniformly on $M \times [0, \infty[$, since each differentiation in x or in y essentially gives a factor n or m, and each differentiation in t gives a factor λ_{mn}. Thus the differential equation and the remaining initial condition can be verified. □

The conditions on f and g in the theorem are *sufficient conditions* — one can weaken them a little and still get solvability — but at least they give some firm ground for the claim that the described procedure gives a solution of the problem posed. (For example, in Theorem 2.3 2°, one can allow the fourth-order derivatives of f and the third-order derivatives of g to be piecewise continuous on M.)

In Example 3.7.1, $h(x) = x(1 - x)$ extends to an odd, 2-periodic function h^* whose first derivative is a continuous triangular function, and the second derivative is piecewise constant, having jumps at the period points $2n$, $n \in \mathbb{Z}$. Then the odd, 2-periodic extension of $f(x, y)$ is C^1 with piecewise continuous (in fact piecewise constant) second derivatives. Here Theorem 2.3 1° gives that the series for $u(x, y, t)$ converges uniformly (which is also clear from the formulas), but (2.24) is not satisfied, and the differential equation holds only in a generalized sense. (In a more advanced theory one will find that the solution is smooth in large areas, but that the irregularities in the initial value propagate along characteristic cones when when t increases.)

2.3 The heat equation with initial data on a rectangle.

The solution formulas for the heat problem [A04, page 161] can be checked in a similar way. Here we find, as for the one-dimensional heat equation, that the solution becomes C^∞ as soon as t becomes positive.

Theorem 2.4. 1° *When $f(x,y)$ is C^2 on $M=[0,a]\times[0,b]$, and f is zero on the boundary ∂M, then*

$$\sum_{m,n\in\mathbb{N}} |A_{mn}| < \infty, \tag{2.31}$$

and the series in (13) *converges uniformly on $M\times[0,\infty[$ to a continuous function $u(x,y,t)$ satisfying the boundary condition $u=0$ for $(x,y)\in\partial M$ and the initial condition $u=f$ for $t=0$.*

2° *Assume merely that $f(x,y)$ is square integrable on M. Then the series in* (13) *and all the termwise differentiated series of arbitrarily high order converge uniformly on $M\times[\varepsilon,\infty[$, for any $\varepsilon>0$. In particular, the differential equation* (11) *and the boundary condition* (12) *are verified for $t>0$.*

Proof. Part 1° is shown in the same way as in Theorem 2.3; the expansion coefficients of f are now called A_{mn}, and the hypotheses assure that the odd periodic extension of f is C^1 with piecewise continuous second derivatives, so that (2.31) holds and defines a majorizing series for (13).

For part 2°, we observe that when $t\ge\varepsilon$, then

$$e^{-\lambda_{mn}^2 t} \le e^{-\lambda_{mn}^2\varepsilon} \le e^{-c_1^2\varepsilon(m^2+n^2)}, \tag{2.32}$$

cf. (2.28). For any $j,k,l\ge 0$, application of $\frac{\partial^j}{\partial x^j}\frac{\partial^k}{\partial y^k}\frac{\partial^l}{\partial t^l}$ termwise gives a series

$$\pm\sum_{m,n\in\mathbb{N}} A_{mn}\Big(\frac{m\pi}{a}\Big)^j\Big(\frac{n\pi}{b}\Big)^k\lambda_{mn}^{2l}\,{\sin\atop\cos}\Big(\frac{m\pi}{a}x\Big)\,{\sin\atop\cos}\Big(\frac{n\pi}{b}y\Big)e^{-\lambda_{mn}^2 t}. \tag{2.33}$$

Since the $|A_{mn}|$ are bounded by a constant (cf. (2.16)ff.), this is majorized by a convergent series

$$\sum_{m,n\in\mathbb{N}} c'(m^2+n^2)^{\frac12 j+\frac12 k+l}e^{-c_1^2\varepsilon(m^2+n^2)} < \infty, \tag{2.34}$$

where the convergence follows e.g. since $(m^2+n^2)^{\frac12 j+\frac12 k+l}e^{-c_1^2\varepsilon(m^2+n^2)} \le c''(m^2+n^2)^{-2}$. Thus u is C^∞ for $t>0$. □

When f is merely square integrable, one can say that the initial condition is verified in the sense that the series (13) for $t=0$ converges in the mean to f on M.

2.4 The Poisson equation with zero boundary data.

For the solution of the Poisson equation on a rectangle put forward in [A04, Section 3.9], sufficient conditions for convergence can likewise be found from Theorem 2.1 ff. The discussion goes rather similarly to that in Theorems 2.3–4:

When the odd periodic extension f^* of f is C^1 with piecewise continuous second derivatives, the sinus coefficients A_{mn} of f satisfy (2.31). Then since $E_{mn} = -A_{mn}/\lambda_{mn}$, one has that

$$\sum_{m,n\in\mathbb{N}} (m^2+n^2)|E_{mn}| < \infty, \tag{2.35}$$

so the differential equation can be verified by termwise differentiation, and $u(x,y)$ is indeed a solution of the problem; it lies in $C^2(M)$.

The solution found in Example 3.9.1 is a generalized solution, and the calculations in Example 3.9.2 are quite formal (the series for u does not satisfy our criteria for termwise differentiation of order 2).

In more advanced treatments of the various differential equations, the theory of Sobolev spaces provides a more satisfactory framework than the spaces of C^l-functions.

2.5 Convergence analysis for the Laplace equation on a disk.

Consider the series solution established in [A04, Section 4.4] for the Laplace equation on a disk with radius a, with a prescribed boundary value f. We shall show that when f is merely square integrable, the series and all termwise differentiated series converge uniformly on the disks with the same center and radius $< a$. For simplicity in the formulation, let us take $a = 1$ and leave the scaling to the general case to the reader.

When $f(\theta)$ is square integrable on $[0, 2\pi]$, its Fourier coefficients are bounded:

$$|a_n| \le C, \ |b_n| \le C, \ \text{for all } n. \tag{2.36}$$

Let $\varepsilon \in]0,1[$. Then the series for u,

$$u(r,\theta) = a_0 + \sum_{n\in\mathbb{N}} r^n(a_n \cos n\theta + b_n \sin n\theta) \tag{2.37}$$

has the majorizing series, when $r \le 1-\varepsilon$:

$$\sum_{n\ge 0} C(1-\varepsilon)^n. \tag{2.38}$$

We can check termwise derivatives in r and θ, showing that each differentiation essentially gives a factor n, so that the series after k differentiations is majorized by a series

$$\sum_{n\ge 0} C'(1+n)^k(1-\varepsilon)^n. \tag{2.39}$$

Why is this convergent? Apply for example the quotient criterion, or note that $(1-\varepsilon)^n = e^{-sn}$, where $-s = \ln(1-\varepsilon) < 0$; here since the exponential function wins over any polynomial, $(1+n)^k e^{-sn} \le C''(1+n)^{-2}$ (as we have used before).

However, if we treat the (r,θ)-derivatives of u, we still have to worry about how the information carries over to the (x,y)-derivatives (in the original coordinates), especially how things fit together at $r = 0$. But there is another point of view that gives the (x,y)-behavior directly:

When the Fourier series of f is written in the complex form

$$f(\theta) = \sum_{m=-\infty}^{\infty} c_m e^{im\theta}, \tag{2.40}$$

we get the series for u in the complex form, that we can reformulate further:

$$\begin{aligned} u(r,\theta) &= \sum_{m\in\mathbb{Z}} r^{|m|} c_m e^{im\theta} \\ &= \sum_{m\geq 0} c_m (re^{i\theta})^m + \sum_{m'>0} c_{-m'}(re^{-i\theta})^{m'} \\ &= \sum_{m\geq 0} c_m (x+iy)^m + \sum_{m'>0} c_{-m'}(x-iy)^{m'}, \end{aligned} \tag{2.41}$$

with bounded coefficients. Here, when (x, y) lies in the disk with radius $1-\varepsilon$, $x+iy$ and $x-iy$ both have absolute value $\leq 1-\varepsilon$. Termwise differentiations in x and y give polynomials in m resp. m' as factors. Then we can again use series of the type in (2.39) as majorizing series, and find that all termwise derived series are uniformly convergent on the smaller disk. Thus u is C^∞ there and satisfies $\Delta u = 0$.

E2. Exercises

Exercise E2.1. Consider the cosine-sine formulation of the Fourier expansion of a function $f(x_1, x_2)$ that is 2π-periodic in each variable. You are asked to express the coefficients of the functions $\cos n_1x_1 \cos n_2x_2$ and $\cos n_1x_1 \sin n_2x_2$ in terms of the coefficients $c_{\mathbf{n}}$ in the series in (2.13). What is the constant term?

Exercise E2.2. Answer Exercise 3.8.12 in [A04], with the additional point:
(d) Show that the differential equation is verified for $z < c$, when $f(x, y)$ is square integrable.

Exercise E2.3. In the following, we identify $\mathbb{Z}^2$ with a subset of $\mathbb{R}^2$, namely with the points with integer coordinates (n_1, n_2).
(a) For each $l \in \mathbb{N}$, show that there are $8l$ of these integer points (n_1, n_2) on the boundary of the square $[-l, l] \times [-l, l]$, and that they satisfy

$$n_1^2 + n_2^2 \geq l^2.$$

(b) Show that there is a constant c such that

$$\sum_{|n_1|\leq l, |n_2|\leq l, (n_1,n_2)\neq(0,0)} \frac{1}{(n_1^2+n_2^2)^2} \leq \sum_{1\leq j\leq l} \frac{c}{l^3}$$

(c) Show that the series

$$\sum_{(n_1,n_2)\in\mathbb{Z}^2\setminus\{(0,0)\}} \frac{1}{(n_1^2+n_2^2)^2}$$

is convergent.

3. Integral transforms

3.1 On Fourier transformation in one variable.

In modern treatments of the Fourier transform one often introduces a space of functions where all rules work without problems, the Schwartz space $\mathcal{S}(\mathbb{R})$ (named after Laurent Schwartz, who introduced it in his treatise on Distribution Theory):

$$\mathcal{S}(\mathbb{R}) = \{\, f \in C^\infty(\mathbb{R}) \mid x^j f^{(k)}(x) \text{ is bounded for all } j,k \geq 0 \,\}. \tag{3.1}$$

Note that when $f \in \mathcal{S}(\mathbb{R})$, then all the derived expressions $x^j f^{(k)}(x)$ are in $\mathcal{S}(\mathbb{R})$. Moreover, they are all integrable, since

$$\int_{-\infty}^{\infty} |f(x)|\, dx = \int_{-\infty}^{\infty} (1+x^2)^{-1}(1+x^2)|f(x)|\, dx \leq C \int_{-\infty}^{\infty} (1+x^2)^{-1}\, dx < \infty,$$

with $C = \max|(1+x^2)f(x)|$. So we can take the Fourier transform of any of these functions,

$$\hat{f}(\omega) = \tfrac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{-ix\omega}\, dx = \mathcal{F}(f)(\omega), \tag{3.2}$$

and this clearly results in a bounded function:

$$|\hat{f}(\omega)| \leq \tfrac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |f(x)e^{-ix\omega}|\, dx = \tfrac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |f(x)|\, dx. \tag{3.3}$$

Let us denote

$$\tfrac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{+ix\omega}\, dx = \overline{\mathcal{F}}(f)(\omega), \tag{3.4}$$

the conjugate Fourier transform. When applied to functions of ω, $\overline{\mathcal{F}}$, acts as an inverse of the Fourier transform. Indeed, it is accounted for in [A04] that for piecewise C^1 and integrable functions $f(x)$,

$$\lim_{N\to\infty} \tfrac{1}{\sqrt{2\pi}} \int_{-N}^{N} \hat{f}(\omega)e^{+ix\omega}\, d\omega = \tfrac{1}{2}(f(x+) + f(x-)). \tag{3.5}$$

This applies in particular when $f \in \mathcal{S}(\mathbb{R})$, giving back $f(x)$.

We shal show that $\mathcal{F}$ maps $\mathcal{S}(\mathbb{R})$ into $\mathcal{S}(\mathbb{R})$, so that the mention of $\lim_{N\to\infty} \frac{1}{\sqrt{2\pi}} \int_{-N}^{N}$ can be replaced by $\int_{-\infty}^{\infty}$ in (3.5). In fact, we shall see that $\mathcal{F}$ is a bijection from $\mathcal{S}(\mathbb{R})$ to $\mathcal{S}(\mathbb{R})$ with inverse $\mathcal{F}^{-1} = \overline{\mathcal{F}}$.

First, the rule that differentiation goes into multiplication by a polynomial,

$$\mathcal{F}(\frac{d^k}{dx^k} f) = (i\omega)^k \hat{f}(\omega), \tag{3.6}$$

is shown as in [A04, Th. 7.2.2].

Remark 3.1. Everywhere in the statement of Theorem 7.2.2, the words "piecewise smooth" should be replaced by "piecewise smooth and continuous" (what we call PC1C). Otherwise there can be contributions from jumps of f that do not cancel out in the integration by parts in the proof, which is very similar to the proof of our Theorem 1.2. This error has been confirmed by the author Nakhlé Asmar by email correspondence.

For f in our class $\mathcal{S}(\mathbb{R})$, (3.6) implies that $\omega^k \hat{f}(\omega)$ is bounded for all k, cf. (3.3).

Secondly, the rule that multiplication by a polynomial is carried over to a differentiation,

$$\mathcal{F}(x^j f(x)) = (i\frac{d}{d\omega})^j \hat{f}(\omega), \tag{3.7}$$

is easily shown for $f \in \mathcal{S}(\mathbb{R})$ by the proof of Th. 7.2.3, since passing $\frac{d}{d\omega}$ through the integration sign is indeed allowed for these functions. Thus, when $f \in \mathcal{S}(\mathbb{R})$, the Fourier transform $\hat{f}$ is infinitely differentiable and all derivatives are bounded. Finally, using the two rules successively, we see that $\omega^j \frac{d^k}{d\omega^k}\hat{f}$ is well-defined and bounded for all j, k, so $\hat{f} \in \mathcal{S}(\mathbb{R})$.

Note that complex conjugation of (3.2) gives

$$\overline{\mathcal{F}(f)} = \overline{\mathcal{F}}(\overline{f}), \tag{3.8}$$

so since $f \in \mathcal{S}(\mathbb{R})$ implies $\overline{f} \in \mathcal{S}(\mathbb{R})$, we see that also $\overline{\mathcal{F}}$ maps $\mathcal{S}(\mathbb{R})$ into $\mathcal{S}(\mathbb{R})$.

Note moreover that $\overline{\mathcal{F}}f$ can be viewed as the Fourier transform of the reflected function $f(-x)$, or as the reflected version of the Fourier transform (whichever point of view one needs):

$$\overline{\mathcal{F}}(f(x))(\omega) = \mathcal{F}(f(-x))(\omega) = \mathcal{F}(f(x))(-\omega); \tag{3.9}$$

this observation is elaborated in Exercise 7.2.10. Clearly, $f(x) \in \mathcal{S}(\mathbb{R})$ implies $f(-x) \in \mathcal{S}(\mathbb{R})$.

Theorem 3.2. *$\mathcal{F}$ defines a bijection of $\mathcal{S}(\mathbb{R})$ onto itself, with inverse $\mathcal{F}^{-1} = \overline{\mathcal{F}}$.*

Proof. Since an element f of $\mathcal{S}(\mathbb{R})$ is C^1 with f and f' integrable and with $\hat{f}$ integrable, (3.5) shows that $\overline{\mathcal{F}}(\mathcal{F}f) = f$. We also need the information that $\mathcal{F}(\overline{\mathcal{F}}f) = f$. It follows for example by conjugation from

$$\overline{\mathcal{F}\overline{\mathcal{F}}f} = \overline{\mathcal{F}}\mathcal{F}\overline{f} = \overline{f}.$$

Or, one can use (3.9). Then $\mathcal{F}$ and $\overline{\mathcal{F}}$ act as each other's inverses on $\mathcal{S}(\mathbb{R})$. □

Note that for any $a > 0$,

$$e^{-ax^2} \in \mathcal{S}(\mathbb{R}), \tag{3.10}$$

since differentiations of this function just lead to polynomials times e^{-ax^2}, where the exponential function "wins over" any polynomial for $x \to \pm\infty$.

Convolution is well-defined for Schwartz functions f and g, and since $\mathcal{F}(f*g) = \hat{f}\cdot\hat{g}$ is a Schwartz function (use the Leibniz rule for the product), we see that also $f*g = \mathcal{F}^{-1}(\hat{f}\hat{g})$ must be one.

There one more important rule for Fourier transforms, that is mentioned in [A04] in Section 11.3 (page 590), the Parseval-Plancherel theorem:

$$\int_{-\infty}^{\infty} f(x)\bar{g}(x)\,dx = \int_{-\infty}^{\infty} \hat{f}(\omega)\bar{\hat{g}}(\omega)\,d\omega; \quad \int_{-\infty}^{\infty} |f(x)|^2\,dx = \int_{-\infty}^{\infty} |\hat{f}(\omega)|^2\,d\omega. \tag{3.11}$$

The proof given there works very well for Schwartz functions. In fact, the formula can be used to extend the Fourier transform and its inverse to square integrable functions by approximation in square mean by Schwartz functions, and here (3.11) remains valid. Because of this rule, the square integrable functions — and certain derived concepts, e.g. Sobolev spaces — play an important role in the applications of Fourier transforms to PDE.

Remark 3.3. The book [A04] gives some basic ingredients of distribution theory in Section 7.8 with applications in Sections 7.9–7.10. Much of this is instructive and consistent with the advanced theory that would be needed for a rigorous deduction. Just one word of warning: One cannot just divide by ω as in the argumentation for formula 7.8(14). Indeed, $x\cdot\delta_0 = 0$, so the equation $x\cdot u = 1$ has many distribution solutions u. In fact, the function $\frac{1}{x}$ is not acceptable as a distribution (since it is not locally integrable), but there is a certain distribution called $\mathrm{PV}\,\frac{1}{x}$ serving its purpose. (For completeness, let us mention that PV stands for "principal value" and refers to the fact that the number that comes out of applying $\mathrm{PV}\,\frac{1}{x}$ to a test function f is defined to be

$$\langle \mathrm{PV}\,\frac{1}{x}, f\rangle = \lim_{\varepsilon\to 0}\int_{|x|\geq\varepsilon} \frac{f(x)}{x}\,dx, \tag{3.12}$$

which exists since f can be written as $f(x) = f(0) + xf_1(x)$ with a C^∞ function f_1.) The Fourier transform of the Heaviside function is not $-\frac{i}{\sqrt{2\pi}}\,\mathrm{PV}\,\frac{1}{\omega}$, as indicated in 7.8(14); it is

$$\mathcal{F}(\mathcal{U}_0) = -\tfrac{i}{\sqrt{2\pi}}\,\mathrm{PV}\,\tfrac{1}{\omega} + \sqrt{\tfrac{\pi}{2}}\,\delta_0. \tag{3.13}$$

It is the sign function $\operatorname{sgn} x$ that has a Fourier transform as in 7.8(14), namely

$$\mathcal{F}(\operatorname{sgn}(x)) = -i\sqrt{\tfrac{2}{\pi}}\,\mathrm{PV}\,\tfrac{1}{\omega}; \tag{3.14}$$

note that $\operatorname{sgn} x = 2\mathcal{U}_0 - 1$. This information on the sign function is consistent with the solution indicated for Exercise 7.8.31, and formula 27 on page A67.

3.2 Multiple Fourier transformations.

We can introduce the Fourier transformation for functions of several variables simply by applying the one-dimensional definition with respect to each variable. Again it is convenient to formulate things for the appropriate Schwartz space and make generalizations afterwards.

Here we define, denoting $\frac{\partial}{\partial x_j} = \partial_j$,

$$\begin{aligned}(3.15)\quad & \mathcal{S}(\mathbb{R}^n) = \\ & \{\, f \in C^\infty(\mathbb{R}^n) \mid x_1^{\alpha_1} \dots x_n^{\alpha_n} \partial_1^{\beta_1} \dots \partial_n^{\beta_n} f(x_1, \dots, x_n) \text{ is bounded for all indices } \geq 0 \,\}.\end{aligned}$$

On this space we define the Fourier transform $\mathcal{F}$ by

$$\begin{aligned}\hat{f}(\omega) &= \tfrac{1}{(2\pi)^{n/2}} \int_{-\infty}^{\infty} e^{-ix_n\omega_n} \cdots \int_{-\infty}^{\infty} e^{-ix_2\omega_2} \int_{-\infty}^{\infty} e^{-ix_1\omega_1} f(x_1, \dots, x_n)\, dx_1 \dots dx_n \\ &= \tfrac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-ix\cdot\omega} f(x)\, dx = \mathcal{F}(f)(\omega),\end{aligned} \tag{3.16}$$

with the usual notation for $\mathbb{R}^n$ ($x \cdot \omega = x_1\omega_1 + \dots x_n\omega_n$, $|x| = (x \cdot x)^{\frac{1}{2}}$). The inversion formula can be checked using the one-dimensional case for each variable, and we get again $\mathcal{F}^{-1} = \overline{\mathcal{F}}$, where

$$\overline{\mathcal{F}}(f)(\omega) = \tfrac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(x) e^{+ix\cdot\omega}\, dx. \tag{3.17}$$

As usual, there are rules for differentiation and multiplication. Consider for example the case $n = 3$, here they are:

$$\begin{aligned}\mathcal{F}(\partial_1^{\alpha_1}\partial_2^{\alpha_2}\partial_3^{\alpha_3} f) &= i^{\alpha_1+\alpha_2+\alpha_3} \omega_1^{\alpha_1}\omega_2^{\alpha_2}\omega_3^{\alpha_3} \hat{f}(\omega), \\ \mathcal{F}(x_1^{\alpha_1}x_2^{\alpha_2}x_3^{\alpha_3} f) &= i^{\alpha_1+\alpha_2+\alpha_3} \partial_1^{\alpha_1}\partial_2^{\alpha_2}\partial_3^{\alpha_3} \hat{f}(\omega);\end{aligned}$$

they are useful in the proof that $\mathcal{F}$ maps the Schwartz space bijectively to itself with inverse $\mathcal{F}^{-1} = \overline{\mathcal{F}}$. For general n it is more convenient to use the multi-index notation:

$$\begin{aligned}&\alpha = (\alpha_1, \dots, \alpha_n), \quad \partial = (\partial_1, \dots, \partial_n), \\ &\partial^\alpha = \partial_1^{\alpha_1} \dots \partial_n^{\alpha_n}, \quad x^\alpha = x_1^{\alpha} \dots x_n^{\alpha_n}, \quad |\alpha| = \alpha_1 + \dots + \alpha_n.\end{aligned} \tag{3.18}$$

Then the rules become

$$\mathcal{F}(x^\alpha \partial^\beta f) = i^{|\alpha|+|\beta|} \partial^\alpha (\omega^\beta \hat{f}(\omega)). \tag{3.19}$$

Let us state the bijectiveness result for the n-dimensional case:

Theorem 3.4. *$\mathcal{F}$ defines a bijection of $\mathcal{S}(\mathbb{R}^n)$ onto itself, with inverse $\mathcal{F}^{-1} = \overline{\mathcal{F}}$.*

The Gaussian functions are defined in n variables as

$$e^{-a|x|^2} = e^{-a(x_1^2+\dots+x_n^2)} = e^{-ax_1^2} \dots e^{-ax_n^2}, \tag{3.20}$$

with $a > 0$. Since such a function is simply a product of one-dimensional Gaussian functions, the transformation rule follows straightforwardly from the rule in one variable:

$$\mathcal{F}(e^{-a|x|^2}) = \frac{1}{(2a)^{n/2}} e^{-|\omega|^2/4a}. \tag{3.21}$$

We also have that

$$\overline{\mathcal{F}}(e^{-a|x|^2}) = \frac{1}{(2a)^{n/2}} e^{-|\omega|^2/4a}, \text{ and } \int_{\mathbb{R}^n} e^{-|x|^2}\, dx = \pi^{n/2}. \tag{3.22}$$

The definition of convolution extends to functions on $\mathbb{R}^n$. Consistently with [A04], we shall here write it

$$(f * g)(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} f(x-y) g(y)\, dy, \tag{3.23}$$

although the convention with a factor $(2\pi)^{-n/2}$ is non-standard. One finds by application of the rule in one dimension to each variable:

$$\mathcal{F}(f * g) = \hat{f} \cdot \hat{g}. \tag{3.24}$$

Now the definition of the Fourier transform $\mathcal{F}$ in (3.12), and the conjugate Fourier transform $\overline{\mathcal{F}}$, make good sense also for functions f that are just integrable on $\mathbb{R}^n$, and (3.3) extends:

$$|\hat{f}(\omega)| \le \tfrac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} |f(x) e^{-ix\cdot\omega}|\, dx = \tfrac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} |f(x)|\, dx. \tag{3.25}$$

The rule that $f = \overline{\mathcal{F}}(\hat{f})$ holds for such functions in a generalized sense (that we shall not explain further here, but just mention for completeness: in the distribution sense), and it holds more concretely under additional hypotheses on f. One also has the Parseval-Plancherel theorem:

$$\int_{\mathbb{R}^n} f(x)\bar{g}(x)\, dx = \int_{\mathbb{R}^n} \hat{f}(\omega)\bar{\hat{g}}(\omega)\, d\omega, \quad \int_{\mathbb{R}^n} |f(x)|^2\, dx = \int_{\mathbb{R}^n} |\hat{f}(\omega)|^2\, d\omega, \tag{3.26}$$

which is easy to verify for functions in $\mathcal{S}(\mathbb{R}^n)$, and which allows an extension of the definition of the Fourier transform to functions that are square integrable on $\mathbb{R}^n$, by approximation by Schwartz functions.

The rule (3.19) extends to “sufficiently regular” functions. Using distribution theory, one can extend the rule to very general situations (including all integrable or square integrable functions).

In the following we shall give some applications of the Fourier theory to PDE, giving proofs in some cases where they are manageable on the present basis, and giving information and references in other cases.

3.3 The n-dimensional heat equation.

The n-dimensional heat problem

$$\begin{aligned}\frac{\partial}{\partial t}u(x,t) &= c^2\Delta_x u(x,t), \quad x\in\mathbb{R}^n,\ t>0,\\ u(x,0) &= f(x), \quad x\in\mathbb{R}^n,\end{aligned} \tag{3.27}$$

is solved just as easily as the one-dimensional problem, thanks to the nice properties of Gaussian functions. Fourier transformation in the x-variable carries this over to the problem

$$\begin{aligned}\frac{\partial}{\partial t}\hat{u}(\omega,t) &= -c^2|\omega|^2\hat{u}(\omega,t), \quad \omega\in\mathbb{R}^n,\ t>0,\\ \hat{u}(\omega,0) &= \hat{f}(\omega), \quad \omega\in\mathbb{R}^n,\end{aligned} \tag{3.28}$$

which we solve for each fixed ω. This gives

$$\hat{u}(\omega,t) = \hat{f}(\omega)e^{-c^2|\omega|^2 t},$$

and hence

$$u(x,t) = f(x) * g(x,t) \text{ for } t>0,$$

where

$$g(x,t) = \mathcal{F}^{-1}(e^{-c^2|\omega|^2 t}) = (2c^2t)^{-n/2}e^{-|x|^2/(4c^2t)},$$

cf. (3.22). Recalling the convention (3.23) for the convolution, we find the formula:

$$\begin{aligned}u(x,t) &= (4\pi c^2t)^{-n/2}\int_{\mathbb{R}^n} f(y)e^{-|x-y|^2/(4c^2t)}\,dy\\ &= (4\pi c^2t)^{-n/2}\int_{\mathbb{R}^n} f(x-y)e^{-|y|^2/(4c^2t)}\,dy.\end{aligned} \tag{3.29}$$

The cases $n=2$ and $n=3$ are of course the most physically interesting.

The deduction makes good sense for functions $f\in\mathcal{S}(\mathbb{R}^n)$, but once we have the formula (3.29), we can apply it to more general functions, getting a solution also then. For example, if f is bounded and continuous — or just piecewise continuous as defined in Remark 2.2 — the integral (3.29) is well-defined for $t>0$. In the first line of (3.29), differentiations can be carried under the integral sign to show that the function satisfies the heat equation. The initial condition can be verified as follows, using the second line in (3.29):

Theorem 3.5. *Let f be bounded and piecewise continuous on $\mathbb{R}^n$, and define $u(x,t)$ by the integral* (3.29)*. Then $u(x,t)\to f(x)$ for $t\to 0$, when x is a point where f is continuous.*

Proof. (It is recommended to go through the proof with $n=1$ in a first reading.)

Consider a point x where f is continuous. We have to show that for any $\varepsilon>0$ there is a $t_0>0$ such that $|u(x,t)-f(x)|\le\varepsilon$ for $0<t\le t_0$. Let M be a constant such that $|f(x)|\le M$ on $\mathbb{R}^n$.

The basic information we shall use is the last equation in (3.22), which gives by scaling, with $v = \frac{1}{2ct^{\frac{1}{2}}} y$:

$$(4\pi c^2 t)^{-n/2} \int_{\mathbb{R}^n} e^{-|y|^2/(4c^2 t)}\, dy = \pi^{-n/2} \int_{\mathbb{R}^n} e^{-|v|^2}\, dv = 1. \tag{3.30}$$

We multiply this complicated formula for 1 with $f(x)$ and subtract this from $u(x,t)$, getting

$$u(x,t) - f(x) = (4\pi c^2 t)^{-n/2} \int_{\mathbb{R}^n} (f(x-y) - f(x)) e^{-|y|^2/(4c^2 t)}\, dy. \tag{3.31}$$

Then, with a similar scaling as in (3.30),

$$\begin{aligned} |u(x,t) - f(x)| &\le (4\pi c^2 t)^{-n/2} \int_{\mathbb{R}^n} |(f(x-y) - f(x))| e^{-|y|^2/(4c^2 t)}\, dy \\ &= \pi^{-n/2} \int_{\mathbb{R}^n} |f(x - 2ct^{\frac{1}{2}} v) - f(x)| e^{-|v|^2}\, dv \\ &= I_N + I'_N; \text{ with} \\ I_N &= \pi^{-n/2} \int_{|v| \le N} |f(x - 2ct^{\frac{1}{2}} v) - f(x)| e^{-|v|^2}\, dv, \\ I'_N &= \pi^{-n/2} \int_{|v| \ge N} |f(x - 2ct^{\frac{1}{2}} v) - f(x)| e^{-|v|^2}\, dv. \end{aligned} \tag{3.32}$$

For a given ε, consider first I'_N. It satisfies

$$I'_N \le 2M\pi^{-n/2} \int_{|v| \ge N} e^{-|v|^2}\, dv \to 0 \text{ for } N \to \infty, \tag{3.33}$$

by definition of the convergent integral. Then we choose N so large that $I'_N \le \varepsilon/2$, and keep N fixed in the sequel. Next, consider I_N; it is estimated as follows:

$$\begin{aligned} I_N &= \pi^{-n/2} \int_{|v| \le N} |f(x - 2ct^{\frac{1}{2}} v) - f(x)| e^{-|v|^2}\, dv \\ &\le \sup_{|x'| \le 2ct^{\frac{1}{2}} N} |f(x - x') - f(x')| \pi^{-n/2} \int_{v \in \mathbb{R}^n} e^{-|v|^2}\, dv = \sup_{|x'| \le 2ct^{\frac{1}{2}} N} |f(x - x') - f(x')|. \end{aligned} \tag{3.34}$$

The continuity of f at x means that for any $\varepsilon' > 0$ there is a $\delta > 0$ such that

$$|f(x - x') - f(x)| \le \varepsilon' \text{ when } |x'| \le \delta. \tag{3.35}$$

Take $\varepsilon' = \varepsilon/2$ and take a δ that satisfies (3.35), and now fix t_0 by

$$t_0 = (\delta/2cN)^2. \tag{3.36}$$

Then when $t \le t_0$, the x' satisfying $|x'| \le 2ct^{\frac{1}{2}} N$ also satisfy $|x'| \le \delta$. It follows from (3.35) that

$$I_N \le \varepsilon/2 \text{ for } t \in]0, t_0]. \tag{3.37}$$

Hence $I_N + I'_N \le \varepsilon$ for such t, which completes the proof. $\square$

3.4 The Laplace equation in a half-space or a ball.

Consider the Dirichlet problem

$$\begin{aligned} &\frac{\partial^2}{\partial x^2}u(x,y)+\frac{\partial^2}{\partial y^2}u(x,y)=0, \quad x\in\mathbb{R},\ y>0,\\ &u(x,0)=f(x), \quad x\in\mathbb{R}. \end{aligned} \tag{3.38}$$

The argumentation in [A04, Sect. 7.5] shows that by the Fourier transform method, discarding unbounded solutions, one obtains Poisson's formula for a solution:

$$\begin{aligned} u(x,y)&=\frac{y}{\pi}\int_{-\infty}^{\infty}\frac{f(s)}{(x-s)^2+y^2}\,ds\\ &=\frac{y}{\pi}\int_{-\infty}^{\infty}\frac{f(x-s)}{s^2+y^2}\,ds. \end{aligned} \tag{3.39}$$

As in the case of the heat equation, one can insert quite general functions f in the formula and get solutions, for example bounded piecewise continuous functions f. When checking that the equation $\Delta u=0$ holds for $t>0$, one uses the first line in (3.39), whereas the second line is useful in a proof that $u(x,y)\to f(x)$ for $y\to 0$ at a point of continuity for f. The proof can be formulated very similarly to Theorem 3.5 and is left to Exercise E3.1.

The solution of (3.38) is not unique without a suitable boundedness condition. There are many unbounded solutions; for example, one can add any of the functions

$$v_1(x,y)=y,\quad v_2(x,y)=xy,\quad v_3(x,y)=\sin x\sinh y,\ \text{or } v_4(x,y)=\sinh x\sin y, \tag{3.40}$$

to u and still have a solution of (3.38). Precise statements on unique solvability can be made in terms of general function spaces (such as Sobolev spaces) that belong to the more advanced theory.

In higher dimensions, there are similar formulas for solutions of the Dirichlet problem

$$\begin{aligned} &\frac{\partial^2}{\partial x_1^2}u(x)+\cdots+\frac{\partial^2}{\partial x_n^2}u(x)=0, \quad x_1,\dots x_{n-1}\in\mathbb{R},\ x_n>0,\\ &u(x_1,\dots,x_{n-1},0)=f(x_1,\dots,x_{n-1}), \quad x_1,\dots,x_{n-1}\in\mathbb{R}. \end{aligned} \tag{3.41}$$

Denote $(x_1,\dots,x_{n-1})=x'$, and denote the Fourier transformation with respect to the x'-variable by

$$\hat u(\omega',x_n)=\mathcal{F}_{x'\to\omega'}u(x',x_n).$$

Then we can write the problem as

$$\begin{aligned} &-|\omega'|^2\hat u(\omega',x_n)+\frac{\partial^2}{\partial x_n^2}\hat u(\omega',x_n)=0, \quad \omega'\in\mathbb{R}^{n-1},\ x_n>0,\\ &\hat u(\omega',0)=\hat f(\omega'), \end{aligned} \tag{3.42}$$

when the functions allow Fourier transformation. This is solved by

$$\hat u(\omega',0)=\hat f(\omega')e^{-|\omega'|x_n}, \tag{3.43}$$

which is the unique solution at each ω', if we require boundedness for $x_n \to \infty$. One can show that for $x_n > 0$,

$$(2\pi)^{-n/2}\mathcal{F}^{-1}_{\omega'\to x'}e^{-|\omega'|x_n} = \frac{2}{n\alpha(n)}\frac{1}{(|x'|^2+x_n^2)^{n/2}}, \tag{3.44}$$

where $\alpha(n)$ denotes the volume of the unit ball in $\mathbb{R}^n$:

$$\alpha(n) = \frac{\pi^{n/2}}{\Gamma(\frac{n}{2}+1)}. \tag{3.45}$$

see for example Strichartz' book on distribution theory [S94, Example 4.2.4]. Then we get the Poisson integral formula for the solution:

$$u(x', x_n) = \frac{2x_n}{n\alpha(n)}\int_{\mathbb{R}^{n-1}}\frac{f(y')}{((x'-y')^2+x_n^2)^{n/2}}\,dy' \tag{3.46}$$

(and the similar formula with y' and $x'-y'$ interchanged). Another proof of (3.46) can be found in Evans' book on PDE [E98, Section 2.2] (and it enters of course in many other books on PDE).

The solution formula for the Dirichlet problem for the Laplace equation on a disk can also be written as an integral formula (likewise called Poisson's formula), see Exercises 4.4.28–29 in [A04].

This formula has a generalization to higher dimensions too. The solution $u(x)$ of the problem for the ball of radius r in $\mathbb{R}^n$ centered at 0,

$$\begin{aligned} \Delta u(x) &= 0, \quad |x| < r,\\ u(x) &= f(x), \quad |x| = r, \end{aligned} \tag{3.47}$$

is:

$$u(x) = \frac{r^2-|x|^2}{n\alpha(n)r}\int_{|y|=r}\frac{f(y)}{|x-y|^n}\,dS(y), \tag{3.48}$$

where $dS(y)$ stand for the measure on the sphere $\{y \in \mathbb{R}^n \mid |y| = r\}$ that is used in polar coordinates. A proof can for example be found in [E98, Sect. 2.2].

There is another equation related to Lapace's equation that is worth mentioning here, the Helmholtz equation

$$(m^2 - \Delta)u(x) = f(x), \quad x \in \mathbb{R}^n, \tag{3.49}$$

$m > 0$. It is very easy to solve using Fourier transformation. In fact, Fourier transformation of (3.49) gives

$$(m^2 + |\omega|^2)\hat{u}(\omega) = \hat{f}(\omega), \tag{3.50}$$

which has the *unique* solution

$$\hat{u}(\omega) = (m^2 + |\omega|^2)^{-1} \hat{f}(\omega), \tag{3.51}$$

hence

$$u = \mathcal{F}^{-1}(\tfrac{1}{m^2+|\omega|^2}\mathcal{F}f) = \mathcal{F}^{-1}(\tfrac{1}{m^2+|\omega|^2}) * f. \tag{3.52}$$

Here, when $f \in \mathcal{S}(\mathbb{R}^n)$, it is easily checked that $\hat{u}$ and hence u is in $\mathcal{S}(\mathbb{R}^n)$, but the formula can be used in much more general situations.

The case of (3.49) with $m = 0$ is much harder to discuss; this is linked with the fact that $\Delta u = 0$ has many solutions on $\mathbb{R}^n$ (harmonic functions), recall for example (3.40).

We shall not bother to give a general formula for $\mathcal{F}^{-1}\frac{1}{m^2+|\omega|^2}$, but will just mention that

$$\begin{aligned} \mathcal{F}^{-1}\frac{1}{m^2+|\omega|^2} &= \frac{\sqrt{2\pi}}{2m} e^{-m|x|}, \text{ when } n = 1, \\ \mathcal{F}^{-1}\frac{1}{m^2+|\omega|^2} &= \frac{\sqrt{2\pi}}{2m^2|x|} e^{-m|x|}, \text{ when } n = 3; \end{aligned} \tag{3.53}$$

the first formula is as in [A04, (7) on page A56]. The formulas in other dimensions involve a Bessel function; more on this in Laurent Schwartz' book on distributions [S66, Ch. VII] (first published in 1959).

3.5 The higher dimensional wave equation.

Whereas the Fourier transformation applied to the 1-dimensional wave equation just gives the d'Alembert formula known from other methods, it plays a more important role in the study of higher dimensional wave equations. We shall here consider the 3-dimensional case. The initial value problem

$$\begin{aligned} \frac{\partial^2}{\partial t^2} u(x,t) &= c^2 \Delta_x u(x,t), \quad x \in \mathbb{R}^3,\ t > 0, \\ u(x,0) &= f(x), \quad x \in \mathbb{R}^3, \\ u_t(x,0) &= g(x), \quad x \in \mathbb{R}^3, \end{aligned} \tag{3.54}$$

gives by Fourier transformation in x

$$\begin{aligned} \frac{\partial^2}{\partial t^2} \hat{u}(\omega,t) &= -c^2|\omega|^2 \hat{u}(\omega,t), \quad \omega \in \mathbb{R}^3,\ t > 0, \\ \hat{u}(\omega,0) &= \hat{f}(\omega), \\ \hat{u}_t(\omega,0) &= \hat{g}(\omega), \end{aligned} \tag{3.55}$$

which we solve for each fixed ω, obtaining

$$\hat{u}(\omega,t) = \hat{f}(\omega) \cos c|\omega|t + \hat{g}(\omega) \frac{\sin c|\omega|t}{c|\omega|}. \tag{3.56}$$

The inverse Fourier transform

$$h(x,t) = \mathcal{F}^{-1}\big(\frac{\sin c|\omega|t}{c|\omega|}\big) \tag{3.57}$$

is worked out for example in [S94]. Since $\frac{\partial}{\partial t}\frac{\sin c|\omega|t}{c|\omega|} = \cos c|\omega|t$, we see that the inverse Fourier transform of $\cos c|\omega|t$ is $\frac{\partial}{\partial t}h(x,t)$, so we can write

$$\begin{aligned} u(x,t) &= f(x) * \tfrac{\partial}{\partial t}h(x,t) + g(x) * h(x,t) \\ &= \tfrac{\partial}{\partial t}(f(x) * h(x,t)) + g(x) * h(x,t). \end{aligned}$$

The resulting formula is

$$u(x,t) = \frac{\partial}{\partial t}\Big(\frac{1}{4\pi c^2 t}\int_{|y|=ct} f(x+y)\,dS(y)\Big) + \frac{1}{4\pi c^2 t}\int_{|y|=ct} g(x+y)\,dS(y), \tag{3.58}$$

known as Kirchhoff's formula (actually due to Poisson). One can deduce from this a formula for the case $n = 2$ by considering functions of (x_1, x_2, x_3, t) that are constant in x_3 (the method of descent).

A proof attacking (3.53) directly is given in Weinberger's classical textbook [W65, §72], other formulations are found in Evans [E98, Sect. 2.2] and Strauss [S92, Sect. 9.2].

E3. Exercises

Exercise E3.1. Consider $u(x,y)$ defined by (3.39) for $y > 0$, with $f(x)$ bounded and piecewise continuous on $\mathbb{R}$. Show that for the points x where f is continuous, one has that $u(x,y) \to f(x)$ for $y \to 0$.

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