Hilbert spaces

X vector space over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$

$\langle\cdot,\cdot\rangle : X \times X \to \mathbb{K}$ <u>scalar product</u> if for all $x,y,z \in X$, $\lambda \in \mathbb{K}$:

$$\left.\begin{array}{l} \cdot\ \langle x+y, z\rangle = \langle x,z\rangle + \langle y,z\rangle \\ \cdot\ \langle \lambda x, y\rangle = \lambda\langle x,y\rangle \end{array}\right\} \text{linearity in first argument} \quad \Bigg\} \text{"sesquilinear" (bilinear if } \mathbb{K}=\mathbb{R})$$

$\cdot\ \langle x,y\rangle = \overline{\langle y,x\rangle}$

$\cdot\ \langle x,x\rangle \geq 0$, $\langle x,x\rangle = 0 \Leftrightarrow x = 0$ $\leftarrow$ "positive definite"

allows to define angles between vectors + norm = "length"

<u>Cauchy-Schwarz inequality:</u> $|\langle x,y\rangle|^2 \leq \langle x,x\rangle\langle y,y\rangle$

"=" $\Leftrightarrow$ $x = \lambda y$ for some $\lambda \in \mathbb{K}$.

<u>Corollary:</u> $\|x\| := \sqrt{\langle x,x\rangle}$ defines a norm on X

<u>Definition:</u> $(X, \langle\cdot,\cdot\rangle)$ is a <u>Hilbert space</u> if X endowed with the topology from $\|\cdot\|$ is complete, i.e. every Cauchy sequence converges:
In other words, given a sequence $\{x_n\}_{n=1}^{\infty}$ in X s.t. for all $\varepsilon > 0$ there exists N such that for all $n,m \geq N$: $\|x_n - x_m\| < \varepsilon$, then $\|x_n - x\| \xrightarrow{n\to\infty} 0$ for some $x \in X$.

<u>Examples:</u> • $\mathbb{R}^n$, $\mathbb{C}^n$ with the usual scalar product

• For $\Omega \subset \mathbb{R}^n$, $w: \Omega \to [0,\infty)$:

$$L^2(\Omega, m\,dx) = \{ f : \Omega \to \mathbb{K} \text{ "measurable"} : \int_\Omega |f|^2 m\, dx < \infty \}$$

$$\langle f,g\rangle := \int_\Omega f \bar{g}\, m\, dx$$

• For any set A: $\ell^2(A) = \{ f : A \to \mathbb{K} : \sum_{a\in A} |f(a)|^2 := \sup_{\substack{F\subset A \\ \text{finite}}} \sum_{a\in F} |f(a)|^2 < \infty \}$

$$\langle f,g\rangle := \sum_{a\in A} f(a)\overline{g(a)}$$

in particular: $A = \{1,\dots,n\} \Rightarrow \ell^2(A) = \mathbb{K}^n$

$A = \mathbb{N} \Rightarrow \ell^2(A) =$ square-summable sequences

Definition:
- $x, y \in X$ <u>orthogonal</u> $\iff \langle x, y\rangle = 0$.
- $B = \{v_\alpha : \alpha \in A\} \subseteq X$ <u>orthonormal</u> $\iff \langle v_\alpha, v_\beta\rangle = \begin{cases} 1, & \alpha = \beta \\ 0, & \alpha \neq \beta \end{cases}$

e.g. $B_1 = \{e^{inx} : n \in \mathbb{N}\} \subseteq L^2((-\pi,\pi), \frac{dx}{2\pi})$

$B_2 = \{1, \sqrt{2}\cos(nx), \sqrt{2}\sin(nx) : n \in \mathbb{N}\} \subseteq L^2((-\pi,\pi), \frac{dx}{2\pi})$

Given such a B, define the <u>Fourier coefficients</u> of $x \in X$ as

$$\hat{x}(\alpha) := \langle x, v_\alpha\rangle$$

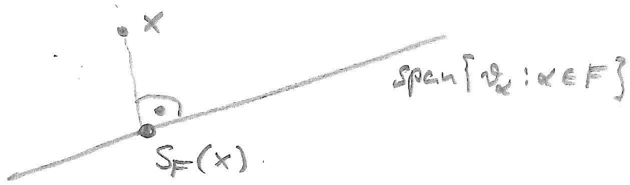
For the above B_1, B_2, these are precisely the real /complex Fourier coefficients that we have been studying.

Further example: $X = \mathbb{K}^n$, $B = \{v_1, \dots, v_n\}$ orthonormal basis

$\Rightarrow \hat{x}(k) = \langle x, v_k\rangle$ = coefficient of x in the expansion with respect to the basis B, i.e. $x = \sum_{k=1}^{n} \hat{x}(k)\, v_k$

$F \subset A$ finite $\Rightarrow$ <u>partial Fourier sum</u> $S_F(x) = \sum_{\alpha \in F} \hat{x}(\alpha)\, v_\alpha$

<u>"Mean square approximation"</u>: $\|x - S_F(x)\| = \min_{u \in \operatorname{span}\{v_\alpha : \alpha \in F\}} \|x - u\|$



<u>Bessel's inequality</u>: $\sum_{\alpha \in A} |\hat{x}(\alpha)|^2 \leq \|x\|^2$

<u>Corollary</u>: The map $X \to \ell^2(A)$, $x \mapsto \hat{x}$ is continuous and surjective.

To see surjectivity: Given $f \in \ell^2(A)$, the vector $x = \sum_{\alpha \in A} f(\alpha)\, v_\alpha$ satisfies $\hat{x} = f$.

Main Theorem 1: The following assertions about an orthonormal set $\mathcal{B} = \{v_\alpha : \alpha \in A\}$ are equivalent:

- $\mathcal{B}$ is maximal, i.e. whenever $\mathcal{B}' \supset \mathcal{B}$ is orthonormal $\Rightarrow \mathcal{B}' = \mathcal{B}$.
- $\overline{\operatorname{span}\{\mathcal{B}\}}^{\|\cdot\|} = X$
- $\langle x, v_\alpha \rangle = 0$ for all $\alpha \in A \Rightarrow x = 0$.
- Parseval 1: $\sum_{\alpha \in A} |\hat{x}(\alpha)|^2 = \|x\|^2$ for all $x \in X$.
- Parseval 2: $\sum_{\alpha \in A} \hat{x}(\alpha)\overline{\hat{y}(\alpha)} = \langle x, y \rangle$ for all $x, y \in X$
- $x \mapsto \hat{x}$ is an isometric isomorphism from X to $\ell^2(A)$.

Such a $\mathcal{B}$ is said to be an orthonormal basis of X.

Zorn's Lemma implies that any orthonormal set can be extended to an orthonormal basis by adding further vectors.

Corollary: Let X be a Hilbert space. Then

1.) X has an orthonormal basis.

2.) There is a set A s.t. $X \cong \ell^2(A)$.

Main Theorem 2: (Measure theory + Weierstrass approximation theorem)

$\mathcal{B}_1 = \{e^{inx} : n \in \mathbb{N}\}$ and $\mathcal{B}_2 = \{1, \sqrt{2}\cos(nx), \sqrt{2}\sin(nx) : n \in \mathbb{N}\}$ are orthonormal bases of $L^2((-\pi,\pi), \frac{dx}{2\pi})$.

Apart from the pointwise and uniform convergence theorems, this includes almost everything we have learned about classical Fourier series.