Geometric singularities and high-order finite elements for the integral fractional Laplacian

Heiko Gimperlein[1]

(with Nikoletta Louca[2], Rafe Mazzeo[3],
Ernst P. Stephan[4], Jakub Stocek[5])

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Set up

Γ Riemannian manifold, $\Omega \subseteq \Gamma$ piecewise smooth domain
$A : H^s(\Gamma) \to H^{-s}(\Gamma)$ elliptic pseudodifferential operator, order $2s$

$$\begin{aligned} Au &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{in } \Gamma \setminus \overline{\Omega}. \end{aligned}$$

Examples:

- $A =$ weakly singular or hypersingular integral operator in BEM
- $A = (-\Delta)^s$ fractional Laplacian, $s \in (0, 1)$

Goals for this talk:

- **Fractional problems interesting**
- **Geometric singularities** of solutions near edges and corners
- **Approximation** by h, p and hp finite elements
- **Exponential convergence** of hp version on geometrically graded meshes

Integral fractional Laplacian

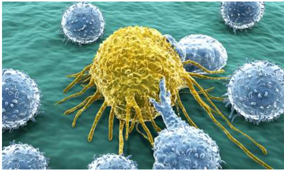
$$(-\Delta)^s u(x) = c_{n,s} \int_{\mathbb{R}^n} \frac{u(x)-u(y)}{|x-y|^{n+2s}} \mathrm{d}y = \mathcal{F}^{-1}(|\xi|^{2s}\mathcal{F}u(x)), \qquad s \in (0,1)$$

- Note: generator of Lévy process and has physical meaning. Used in probability, PDE, applications.
- Recent numerical analysis of $(-\Delta)^s$: Acosta, Ainsworth, Borthagaray, Karkulik, Melenk, Nochetto, Salgado, Schwab . . . (2017 –)
- Recent analysis: Caffarelli, Silvestre, Figalli . . . (2007 –)
- Recent modeling: Du (continuum mechanics, ICM 2018), Perthame (cell movement, 2018), Mouhot (kinetic eqns), . . .
- Don't confuse with spectral $(-\Delta)^s$, the fractional power of the Dirichlet problem (Banjai, Borthagaray, Nochetto, Melenk, Otarola, Salgado, . . .).

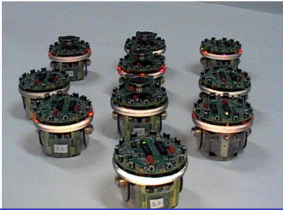
$$(-\Delta)^s u = f \quad \text{in } \Omega,$$
$$u = 0 \quad \text{in } \Gamma \setminus \overline{\Omega}.$$

Classical Applications:
Finance / option pricing: Tankov (2003)
Continuum Mechanics: Du (2018)

Nonlocal movement of cells and organisms
Perthame, Sun, Tang, ZAMP 2018
Estrada-Rodriguez, HG, Painter, SIAP 2018
Estrada-Rodriguez, HG, Painter, Stocek, M3AS 2019
Estrada, Estrada-Rodriguez, HG, SIAM Review 2020
Estrada-Rodriguez, Perthame 2022



Swarm robotic systems
Estrada-Rodriguez, HG, SIAP 2020
Duncan, Dragone, Estrada-Rodriguez, HG, Stocek, Vargas, Bioinsp & Biomim 2022



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- $A = (-\Delta)^s$ fractional Laplacian, $s \in (0, 1)$

For $\Gamma = \mathbb{R}^n$:

$$(-\Delta)^s u(x) = c_{n,s} \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x - y|^{n+2s}} \mathrm{d}y, \qquad s \in (0, 1) \ .$$

Relationship to FEM and BEM

$$(-\Delta)^s u(x) = c_{n,s} \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x - y|^{n+2s}} \mathrm{d}y, \qquad s \in (0, 1) \ .$$

For $s = 1$ the fractional Dirichlet problem

$$\begin{aligned} (-\Delta)^1 u = f \quad & \text{in } \Omega \subset \mathbb{R}^n, \\ u = 0 \quad & \text{in } \mathbb{R}^n \setminus \overline{\Omega}. \end{aligned}$$

is equivalent to the Dirichlet problem for the Laplacian.

This is most easily seen from $(-\Delta)^s u = \mathcal{F}^{-1}(|\xi|^{2s} \mathcal{F} u)$.

Relationship to FEM and BEM

$$(-\Delta)^s u(x) = c_{n,s} \int_{\mathbb{R}^n} \frac{u(x)-u(y)}{|x-y|^{n+2s}} \mathrm{d}y, \qquad s \in (0,1)\ .$$

For $s = \frac{1}{2}$ the fractional Dirichlet problem

$$\begin{aligned} (-\Delta)^{1/2} u = f \quad & \text{in } \Omega \subset \mathbb{R}^n, \\ u = 0 \quad & \text{in } \mathbb{R}^n \setminus \overline{\Omega}. \end{aligned}$$

is equivalent to hypersingular integral equation on **flat screen**:

$$2Wu = f \quad \text{on } \Omega \times \{0\} \subset \mathbb{R}^{n+1}.$$

(see HG, Stocek, Urzúa-Torres, Numer. Math. 2021)

Relationship to FEM and BEM

$$(-\Delta)^s u(x) = c_{n,s} \int_{\mathbb{R}^n} \frac{u(x)-u(y)}{|x-y|^{n+2s}} \mathrm{d}y, \qquad s \in (0,1)\ .$$

For $s = -\frac{1}{2}$ the fractional Dirichlet problem

$$\begin{aligned} (-\Delta)^{-1/2} u &= f \quad \text{in } \Omega \subset \mathbb{R}^n, \\ u &= 0 \quad \text{in } \mathbb{R}^n \setminus \overline{\Omega}. \end{aligned}$$

is equivalent to weakly singular integral equation on **flat screen**:

$$2Vu = f \quad \text{on } \Omega \times \{0\} \subset \mathbb{R}^{n+1}.$$

(see HG, Stocek, Urzúa-Torres, Numer. Math. 2021)

Set up

Fractional Dirichlet problem:

$$(-\Delta)^s u = f \qquad \text{in } \Omega$$
$$u = 0 \qquad \text{in } \mathbb{R}^n \setminus \overline{\Omega}$$

Variational formulation: Find $u \in H$ such that for all $v \in H$

$$E(u) = \frac{c_{n,s}}{4} \iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{(u(x) - u(y))^2}{|x-y|^{n+2s}} \, dy \, dx - \int_\Omega f(x) u(x) \, dx \leq E(v)$$

Sobolev spaces: $H = \widetilde{H}^s(\Omega)$

$$\widetilde{H}^s(\Omega) = \left\{ v \in L^2(\mathbb{R}^n) : \iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{(v(x) - v(y))^2}{|x-y|^{n+2s}} < \infty, v = 0 \text{ in } \mathbb{R}^n \setminus \overline{\Omega} \right\}$$

Lax-Milgram: If $f \in L^2(\Omega) \rightsquigarrow \exists!$ minimizer $u \in H$

Set up

Fractional Dirichlet problem:

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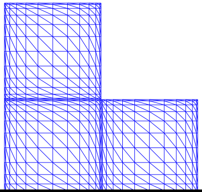
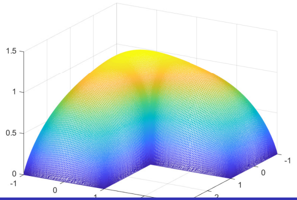
Finite element approximation: Find $u_h \in H_h \subset H$ s.t for all $v_h \in H_h$

$$E(u_h) = \frac{c_{n,s}}{4} \iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{(u_h(x) - u_h(y))^2}{|x-y|^{n+2s}}\, dy\, dx - \int_\Omega f(x) u_h(x)\, dx \;\leq\; E(v_h)$$

$\rightsquigarrow$ linear system of equations





Set up

Fractional Dirichlet problem:

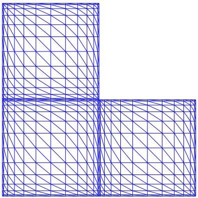
$$\begin{aligned}(-\Delta)^s u = f \qquad & \text{in } \Omega \\ u = 0 \qquad & \text{in } \mathbb{R}^n \setminus \overline{\Omega}\end{aligned}$$

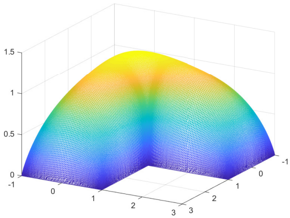
Finite element approximation: Find $u_h \in H_h \subset H$ s.t for all $v_h \in H_h$

$$E(u_h) = \frac{c_{n,s}}{4} \iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{(u_h(x)-u_h(y))^2}{|x-y|^{n+2s}}\, dy\, dx - \int_\Omega f(x) u_h(x)\, dx \ \le \ E(v_h)$$

Lax-Milgram: If $f \in L^2(\Omega) \rightsquigarrow \exists!$ minimizers $u \in H$, $u_h \in H_h$

Finer information about $u \rightsquigarrow$ fast approximation: graded mesh, hp





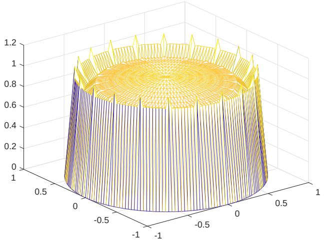
Numerical approximation: non-trivial

$$(-\Delta)^s u = 1 \text{ in } \Omega$$
$$u = 0 \text{ in } \mathbb{R}^2 \setminus \overline{\Omega}.$$

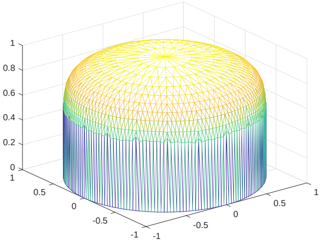
For $\Omega = \mathcal{B}_1 = \{|x| < 1\}$. $s = \frac{1}{10}$

Exact solution: $u(x) = (1 - |x|^2)_+^{\frac{1}{10}}$

Uniform mesh



2-graded mesh



Numerical approximation on graded meshes

$$(-\Delta)^s u = 1 \quad \text{in } \Omega,$$
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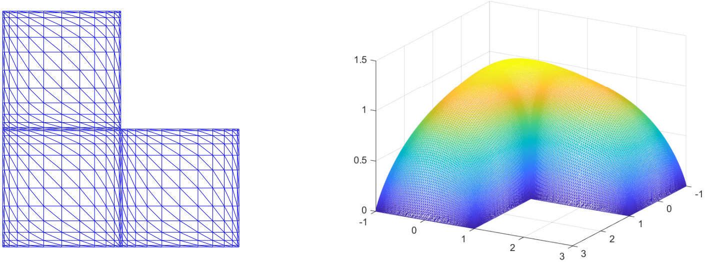
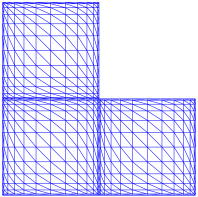


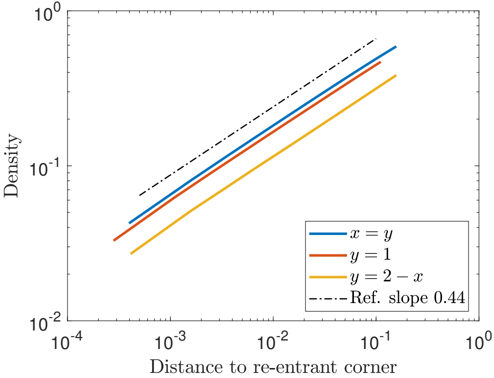
Figure: Algebraically 2–graded mesh for L–shape (left) and numerical solution with $s = \frac{1}{2}$ (right).

Edge and corner singularities

$$(-\Delta)^s u = 1 \quad \text{in } \Omega, \qquad (\star)$$
$$u = 0 \quad \text{in } \mathbb{R}^2 \setminus \overline{\Omega}.$$

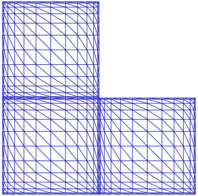
Numerically $u(x) \sim \mathrm{dist}(x, \mathrm{corner})^\lambda$.

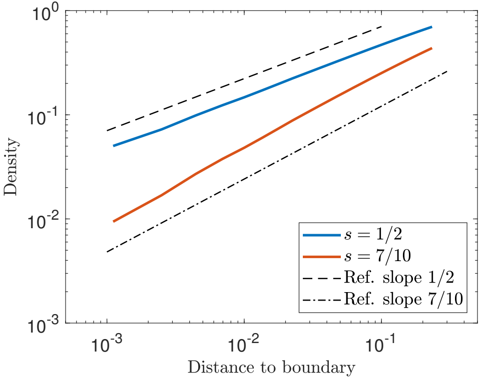




$$\begin{aligned} (-\Delta)^s u &= 1 \quad \text{in } \Omega, \\ u &= 0 \quad \text{in } \mathbb{R}^2 \setminus \overline{\Omega}. \end{aligned} \qquad (\star)$$

Numerically $u(x) \sim \mathrm{dist}(x, \partial\Omega)^s$.





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\tag{$\star$}
$$

We obtain a precise description of the solution:

Theorem (HG, Mazzeo, Louca / HG, Stephan, Stocek 2022)

For Ω polygon and $f \in C^\infty(\overline{\Omega})$, the solution u to $(\star)$ admits an asymptotic expansion at the edges and corners:

Edge E: $u(x) \sim \mathrm{dist}(x, E)^s$,
Corner C: $u(x) \sim \mathrm{dist}(x, C)^\lambda$,
Edge-Corner: $u(x) \sim \mathrm{dist}(x, C)^{\lambda - s} \cdot \mathrm{dist}(x, E)^s$,

up to logarithmic terms.

Here, λ relates to the smallest eigenvalue of an elliptic 2nd order differential operator on S^2_+.

Corner singularity: Dependence on angle and s

$s = \frac{1}{2}$ classical: J. A. Morrison, J. A. Lewis '76, Walden '74.

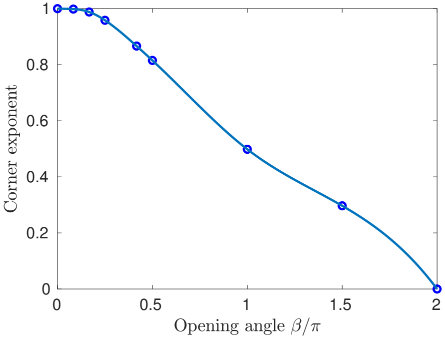

Figure: Corner exponent against opening angle χ, $s = \frac{1}{2}$.

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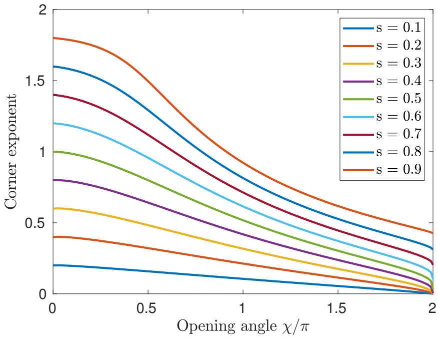


Figure: HG, Stephan, Stocek 2022:
Corner exponent against opening angle χ for different values of s.

Theorem:
$\lambda(s, \chi) > \max\{s - \frac{1}{2}, 0\}$ increasing in s, decreasing in angle χ.

Classical work: Edge behaviour of solutions

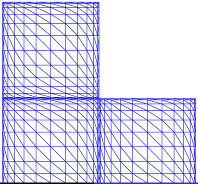
Theorem (Ros-Oton, Serra 2017 / Grubb 2015)

Let $s \neq \frac{1}{2}$, Ω sufficiently smooth, $f \in L^\infty(\Omega)$.
Then $\frac{u(x)}{\mathrm{dist}(x,\partial\Omega)^s} \in C^\alpha$ for some $\alpha > 0$.
Logarithmic corrections for $s = \frac{1}{2}$.

Theorem (Acosta, Borthagaray 2017)

Let Ω convex polyhedron. Then quasi-optimal convergence on $\beta-$graded meshes:

$$\|u - u_h\|_{H^s} \lesssim h^{\min\{\frac{\beta}{2}, 2-s\}-\varepsilon}.$$



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Borthagaray, Nochetto, et al.: Graded meshes using Besov space estimates

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Today:
Geometric singular analysis,
approximation on graded meshes, *hp* and exponential convergence.

Edge and corner singularities

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up to logarithmic terms.

Resolving geometric singularities: Graded meshes & *hp*

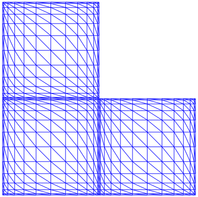
$$-(-\Delta)^s u = f \qquad \text{in } \Omega$$
$$u = 0 \qquad \text{in } \mathbb{R}^n \setminus \overline{\Omega}$$

Theorem (HG, Stephan, Stocek 2022)

Quasi-optimal convergence of finite elements on β-graded meshes:

$$\|u - u_h\|_{\tilde{H}^s} \lesssim h^{\min\{\frac{\beta}{2}, 2-s\}} |\log^*(h)| \ .$$

related work on BEM: von Petersdorff, Stephan 1990



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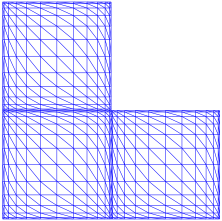

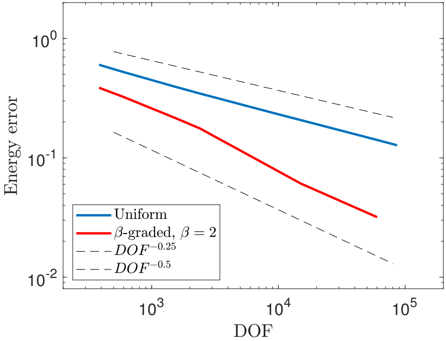

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Theorem (HG, Stephan, Stocek 2022)

Doubled convergence rate of p-version on quasi-uniform mesh:

$$\|u - u_{hp}\|_{\tilde{H}^s} \lesssim \left(\frac{h^{1/2}}{p}\right) |\log^*(h/p^2)| \; .$$

BEM: Bespalov, Heuer 2005 – 2010

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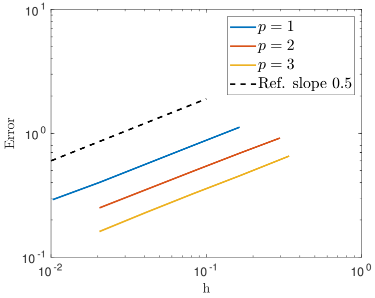

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Exponential convergence of *hp* version

Theorem (HG, Stephan, Stocek 2022)

Exponential convergence of hp finite elements on geometrically graded meshes:

$$\|u - u_h\|_{\tilde{H}^s} \lesssim \exp(-C(DOF)^{1/4}) \ .$$

BEM: Heuer, Maischak, Stephan 1999,
Holm, Maischak, Stephan 2008.

$(-\Delta)^s$ & countably normed spaces:
Faustmann, Marcati, Melenk, Schwab 2022.

Geometrically graded meshes with $\sigma = 0.5, \ 0.17$
$p = 1$ at $\partial\Omega$, linear increasing with elements

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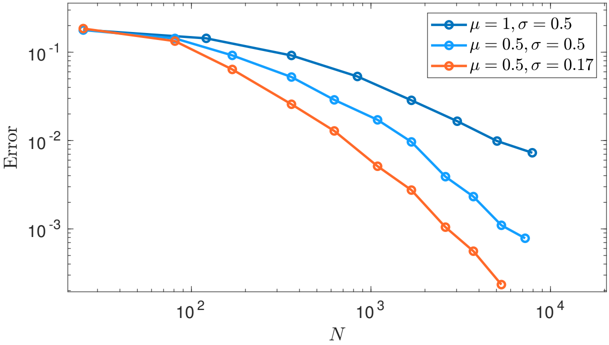
$(-\Delta)^{3/4} u = 1$ in $\Omega = [-1, 1]^2$



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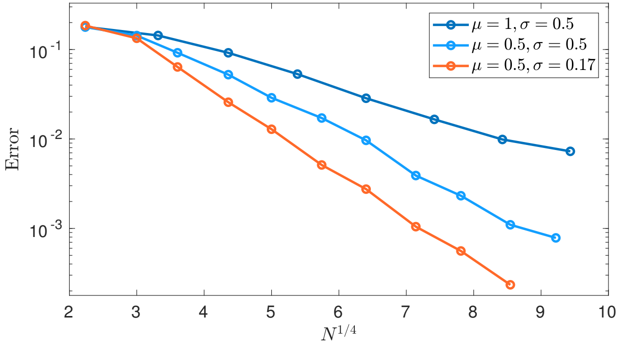

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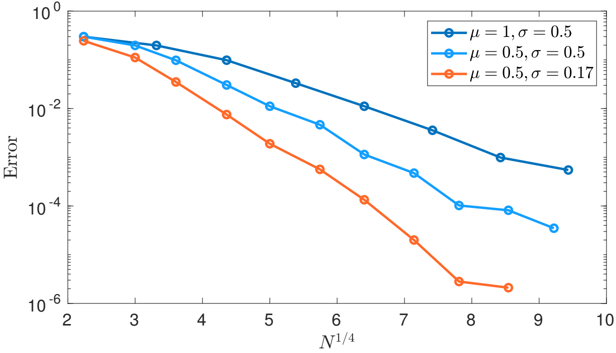

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Doubled convergence rate of p-version on quasi-uniform mesh:

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Theorem (HG, Stephan, Stocek 2022)

Exponential convergence of hp finite elements on geometrically graded meshes:

$$\|u - u_h\|_{\tilde{H}^s} \lesssim \exp(-C(DOF)^{1/4}) \ .$$

Ideas of proof: exponential convergence

Theorem (EPS, Gimperlein, Stocek 2022)

Exponential convergence of hp finite elements on geometrically graded rectangular meshes:

$$\|u - u_h\|_{\tilde{H}^s} \lesssim \exp(-C(DOF)^{1/4}) \ .$$

$(-\Delta)^s$ & countably normed spaces:
Faustmann, Marcati, Melenk, Schwab 2022.

Combine their regularity result with approximation arguments by Maischak (Habilitation 2004), which establish exponential convergence for $u \in B^1_\beta(\Omega)$ (countably normed space of Babuska-Guo).

Ideas of proof: exponential convergence

$(-\Delta)^s$ & countably normed spaces:
Faustmann, Marcati, Melenk, Schwab 2022.

Combine their regularity result with approximation arguments by Maischak (Habilitation 2004), which establish exponential convergence for $u \in B^1_\beta(\Omega)$ (countably normed space of Babuska-Guo).

$$H^{k,1}_\beta(\Omega) = \{u \in L^2(\Omega) : \Phi_{\beta,\alpha,1}\partial^\alpha u \in L^2(\Omega) \forall 1 \leq |\alpha| \leq k\}$$

$u \in B^1_\beta(\Omega)$ if $u \in \bigcap_{k\geq 1} H^{k,1}_\beta(\Omega)$ and there exist $C, d \geq 1$ such that for all $k \geq 1$ and all $|\alpha| = k$:

$$\|\Phi_{\beta,\alpha,1}\partial^\alpha u\|_{L^2(Q)} \leq C d^{k-1}(k-1)!$$

For $[0,1]^2$

$$\Phi_{\beta,(\alpha_1,\alpha_2),1} = \begin{cases} x^{\beta+\alpha_1-1} & \text{for } \alpha_1 \geq 1, \alpha_2 = 0 \\ x^{\beta+\alpha_1-1}y^{\alpha_2} + x_1^{\alpha} y^{\beta+\alpha_2-1} & \text{for } \alpha_1 \geq 1, \alpha_2 \geq 1 \\ y^{\beta+\alpha_2-1} & \text{for } \alpha_1 = 0, \alpha_2 \geq 1. \end{cases}$$

Ideas of proof: exponential convergence

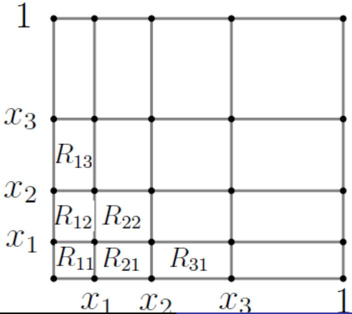
$$H^{k,1}_\beta(\Omega) = \{u \in L^2(\Omega) : \Phi_{\beta,\alpha,1}\partial^\alpha u \in L^2(\Omega) \forall 1 \leq |\alpha| \leq k\}$$

$u \in B^1_\beta(\Omega)$ if $u \in \bigcap_{k\geq 1} H^{k,1}_\beta(\Omega)$ and there exist $C, d \geq 1$ such that for all $k \geq 1$ and all $|\alpha| = k$:

$$\|\Phi_{\beta,\alpha,1}\partial^\alpha u\|_{L^2(Q)} \leq Cd^{k-1}(k-1)!$$

Local approximation for $u \in B^1_\beta(\Omega)$:

interior elements u^A, edge u^B, u^C, Corner u^D



Edge and corner singularities

$$
\begin{aligned}
(-\Delta)^s u &= f \quad \text{in } \Omega, \\
u &= 0 \quad \text{in } \mathbb{R}^2 \setminus \overline{\Omega}.
\end{aligned}
\qquad (\star)
$$

The results on quasi-uniform and alg. graded meshes depend on a precise description of the solution:

Theorem (HG, Mazzeo, Louca / HG, Stephan, Stocek 2022)

For Ω polygon and $f \in C^\infty(\overline{\Omega})$, the solution u to $(\star)$ admits an asymptotic expansion at the edges and corners:

Edge E: $u(x) \sim \mathrm{dist}(x, E)^s$,
Corner C: $u(x) \sim \mathrm{dist}(x, C)^\lambda$,
Edge-Corner: $u(x) \sim \mathrm{dist}(x, C)^{\lambda - s} \cdot \mathrm{dist}(x, E)^s$,

up to logarithmic terms.

Extension approach, $s = 1/2$

Fractional Dirichlet problem:

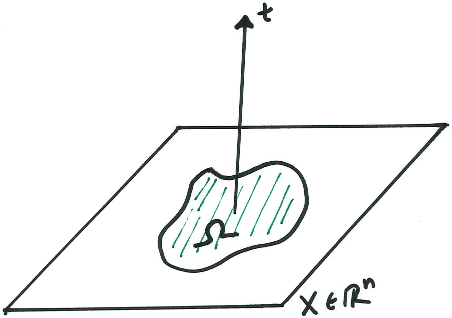
$$\begin{aligned} (-\Delta)^{\frac{1}{2}} u &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{in } \mathbb{R}^n \setminus \overline{\Omega}. \end{aligned} \qquad (\star)$$

Harmonic extension of u:

$$\begin{aligned} \Delta_{(X,t)} U(X,t) &= 0 && \text{for } X \in \mathbb{R}^n \text{ and } t > 0 \\ U(X,0) &= u(X) && \text{for } X \in \mathbb{R}^n \end{aligned}$$

$$U(X,t) \to 0 \quad \text{as } |X,t| \to \infty$$

Extension approach, $s = 1/2$



Fractional Dirichlet problem:

$$\begin{aligned} (-\Delta)^{\frac{1}{2}} u &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{in } \mathbb{R}^n \setminus \overline{\Omega}. \end{aligned} \qquad (\star)$$

Mixed problem:

$$\begin{aligned} \Delta_{(X,t)} U(X,t) &= 0 && \text{for } X \in \mathbb{R}^n \text{ and } t > 0 \\ -\lim_{t\to 0^+} \partial_t U(X,t) &= f(X) && \text{for } X \in \Omega \\ U(X,0) &= 0 && \text{for } X \in \mathbb{R}^n \setminus \overline{\Omega} \end{aligned} \qquad (\star\star)$$

$$U(X,t) \to 0 \quad \text{as } |X,t| \to \infty$$

Extension approach, $s = 1/2$

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Theorem (Caffarelli, Silvestre 2007, $\Omega = \mathbb{R}^n$)

- $(-\Delta)^{1/2}$ *coincides with the Dirichlet-to-Neumann operator:*

$$T : u \mapsto -U_t(x,0).$$

- $(\star) \Leftrightarrow (\star\star) \Leftrightarrow Tu = f$.

Extension approach, $s \neq 1/2$

Fractional Dirichlet problem:

$$\begin{aligned} (-\Delta)^s u &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{in } \mathbb{R}^n \setminus \overline{\Omega}. \end{aligned} \qquad (\star)$$

Extension: (degenerately elliptic for $s \neq 1/2$)

$$\begin{aligned} \nabla_{(X,t)} \cdot (t^{1-2s} \nabla_{(X,t)} U(X,t)) &= 0 && \text{for } X \in \mathbb{R}^n \text{ and } t > 0 \\ U(X,0) &= u(X) && \text{for } X \in \mathbb{R}^n \end{aligned} .$$

$$U(X,t) \to 0 \quad \text{as } |X,t| \to \infty$$

Extension approach, $s \neq 1/2$

Fractional Dirichlet problem:

$$\begin{aligned} (-\Delta)^s u &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{in } \mathbb{R}^n \setminus \overline{\Omega}. \end{aligned} \qquad (\star)$$

Mixed PDE problem:

$$\begin{aligned} \nabla_{(X,t)} \cdot (t^{1-2s} \nabla_{(X,t)} U(X,t)) &= 0 && \text{for } X \in \mathbb{R}^n \text{ and } t > 0 \\ -\lim_{t\to 0^+} t^{1-2s} \partial_t U(X,t) &= f(X) && \text{for } X \in \Omega \\ U(X,0) &= 0 && \text{for } X \in \mathbb{R}^n \setminus \overline{\Omega} \end{aligned} .$$

$$U(X,t) \to 0 \quad \text{as } |X,t| \to \infty \qquad (\star\star\star)$$

Extension approach, $s \neq 1/2$

Fractional Dirichlet problem:

$$\begin{aligned} (-\Delta)^s u &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{in } \mathbb{R}^n \setminus \overline{\Omega}. \end{aligned} \qquad (\star)$$

Mixed PDE problem:

$$\begin{aligned} \nabla_{(X,t)} \cdot (t^{1-2s} \nabla_{(X,t)} U(X,t)) &= 0 && \text{for } X \in \mathbb{R}^n \text{ and } t > 0 \\ -\lim_{t\to 0^+} t^{1-2s} \partial_t U(X,t) &= f(X) && \text{for } X \in \Omega \\ U(X,0) &= 0 && \text{for } X \in \mathbb{R}^n \setminus \overline{\Omega} \end{aligned} .$$

$$U(X,t) \to 0 \quad \text{as } |X,t| \to \infty \qquad (\star\star\star)$$

Theorem (Caffarelli, Silvestre 2007, $\Omega = \mathbb{R}^n$)

- $(-\Delta)^s u = Tu = -c_{n,s} \lim_{t\to 0^+} t^{1-2s} \partial_t U.$
- $(\star) \Leftrightarrow (\star\star\star) \Leftrightarrow Tu = f.$

Key observation: fractional $\Leftrightarrow$ mixed PDE also for $\Omega \subset \mathbb{R}^n$.

Extension approach, $s \neq 1/2$

Fractional Dirichlet problem:

$$\begin{aligned} (-\Delta)^s u &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{in } \mathbb{R}^n \setminus \overline{\Omega}. \end{aligned} \qquad (\star)$$

Mixed PDE problem:

$$\begin{aligned} \nabla_{(X,t)} \cdot (t^{1-2s} \nabla_{(X,t)} U(X,t)) &= 0 && \text{for } X \in \mathbb{R}^n \text{ and } t > 0 \\ -\lim_{t \to 0^+} t^{1-2s} \partial_t U(X,t) &= f(X) && \text{for } X \in \Omega \\ U(X,0) &= 0 && \text{for } X \in \mathbb{R}^n \setminus \overline{\Omega} \end{aligned} .$$

$$U(X,t) \to 0 \quad \text{as } |X,t| \to \infty \qquad (\star\star\star)$$

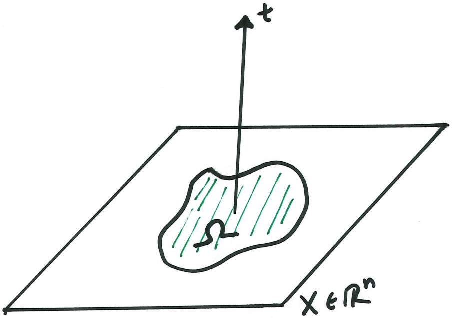
Key observation: fractional $\Leftrightarrow$ mixed PDE also for $\Omega \subset \mathbb{R}^n$.

Consequence: relation to classical problems

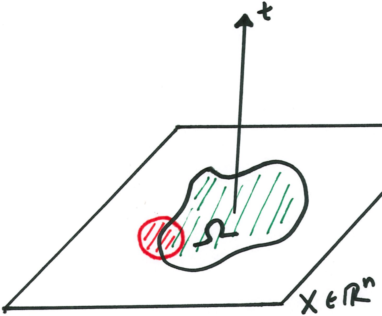
We can use classical techniques for the singular analysis of mixed boundary problems.

Extension approach to fractional boundary problems

$$(-\Delta)^s u = f \quad \text{in } \Omega,$$
$$u = 0 \quad \text{in } \mathbb{R}^n \setminus \overline{\Omega}.$$

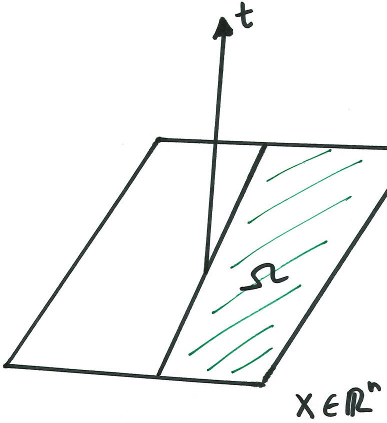


Localisation:



Extension approach to fractional boundary problems

$$
\begin{aligned}
L_s U(X,t) &= 0 && \text{for } X \in \mathbb{R}^n,\ t>0,\\
U(X,0) &= 0 && \text{for } X \in \mathbb{R}^n \setminus \overline{\Omega},\\
-\lim_{t\to 0^+} t^{1-2s}\partial_t U(X,t) &= f(X) && \text{for } X \in \Omega,\ t>0,\\
U(X,t) &\to 0 && \text{as } |X,t| \to \infty.
\end{aligned}
$$



Extension approach to fractional boundary problems

Coordinates in half-space $(X, t) \in \mathbb{R}^{n+1}_+$:
$X = (x, y) \in \mathbb{R} \times \mathbb{R}^{n-1}$, $t > 0$.
Fourier transform tangential to boundary

$$\mathcal{F}_{y \to \eta} L_s = \partial_t^2 + \frac{1-2s}{t}\partial_t + \partial_x^2 - |\eta|^2 =: \widehat{L_s}$$

Mixed boundary problem now becomes

$$\begin{aligned} \widehat{L_s}\widehat{U}(x,\eta,t) &= 0 \quad \text{in } \mathbb{R}^{n+1}_+, \\ \widehat{U}(x,\eta,0) &= 0 \quad \text{for } x<0, \\ -\lim_{t\to 0^+} t^{1-2s}\partial_t \widehat{U}(x,\eta,0) &= \widehat{u}(x,\eta,0) \quad \text{for } x>0. \\ U(X,t) \to 0 \ \text{ as } |X,t| \to \infty \end{aligned} \qquad (\widehat{\mathrm{MBP}})$$

Local behaviour near $\partial\Omega$: $|\eta|^2$ lower order $\to$ Consider $\eta = 0$ first.
Polar coordinates in 2d

$$t = \rho\sin\theta,\ x = \rho\cos\theta$$

$$B(L_s) = \partial_\rho^2 + \frac{2-2s}{\rho}\partial_\rho + \frac{1}{\rho^2}\left(\partial_\theta^2 + (1-2s)\cot(\theta)\partial_\theta\right).$$

Extension approach to fractional boundary problems

$$B\left(L_s\right) = \partial_\rho^2 + \frac{2-2s}{\rho}\partial_\rho + \frac{1}{\rho^2}\underbrace{\left(\partial_\theta^2 + (1-2s)\cot(\theta)\partial_\theta\right)}_{:=P_s}$$

Asymptotics from separation of variables:
$U = \rho^\nu \phi(\theta) \to$ eigenvalue problem:

$$P_s\phi = -\lambda(\nu)\phi$$

with the mixed boundary conditions

$$\theta^{1-2s}\partial_\theta\phi = 0 \quad \text{at } \theta = 0, \qquad \phi = 0 \qquad \text{at } \theta = \pi$$

where $\lambda(\nu) = (\nu + \frac{1}{2})^2 - (s - \frac{1}{2})^2$.

Extension approach to fractional boundary problems

$$B(L_s) = \partial_\rho^2 + \frac{2-2s}{\rho}\partial_\rho + \frac{1}{\rho^2}\underbrace{\left(\partial_\theta^2 + (1-2s)\cot(\theta)\partial_\theta\right)}_{:=P_s}$$

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where $\lambda(\nu) = (\nu + \frac{1}{2})^2 - (s - \frac{1}{2})^2$.

Lemma

P_s is self-adjoint and negative on $L^2\left([0,\pi], \sin^{1-2s}(\theta)\, d\theta\right)$.

Extension approach to fractional boundary problems

$$B(L_s) = \partial_\rho^2 + \frac{2-2s}{\rho}\partial_\rho + \frac{1}{\rho^2}\underbrace{\left(\partial_\theta^2 + (1-2s)\cot(\theta)\partial_\theta\right)}_{:=P_s}$$

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Lemma

P_s is self-adjoint and negative on $L^2\left([0,\pi], \sin^{1-2s}(\theta)\, d\theta\right)$.
Classical (Zaremba) $s = 1/2$:
Eigenvalues: $\lambda_j = \left(j+\frac{1}{2}\right)^2$, eigenfunctions: $\phi_j = \cos\left(\left(j+\frac{1}{2}\right)\theta\right)$,
Singular expansion $u \sim \sum u_j(y)x^{j+\frac{1}{2}}$ with exponents $\nu_j = j + \frac{1}{2}$.

Extension approach to fractional boundary problems

$$B(L_s) = \partial_\rho^2 + \frac{2-2s}{\rho}\partial_\rho + \frac{1}{\rho^2}\underbrace{\left(\partial_\theta^2 + (1-2s)\cot(\theta)\partial_\theta\right)}_{:=P_s}$$

Asymptotics from separation of variables:

$$P_s\phi = -\lambda(\nu)\phi$$

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where $\lambda(\nu) = (\nu + \frac{1}{2})^2 - (s - \frac{1}{2})^2$.

Lemma

P_s is self-adjoint and negative on $L^2\left([0,\pi], \sin^{1-2s}(\theta)\, d\theta\right)$.
$\lambda_j^s = (j+s)(j+1-s)$, $\varphi_j^s = \sin^s(\theta) P_j^s(\cos(\theta))$,
Singular exponents: $\nu_j^s = j + s$.
Here P_j^s is the associated Legendre function of the first kind.

Extension approach to fractional boundary problems

$$B(L_s) = \partial_\rho^2 + \frac{2-2s}{\rho}\partial_\rho + \frac{1}{\rho^2}\underbrace{\left(\partial_\theta^2 + (1-2s)\cot(\theta)\partial_\theta\right)}_{:=P_s}$$

Asymptotics from separation of variables:

$$P_s\phi = -\lambda(\nu)\phi$$

with the mixed boundary conditions

$$\theta^{1-2s}\partial_\theta\phi = 0 \quad \text{at } \theta = 0, \qquad \phi = 0 \qquad \text{at } \theta = \pi$$

where $\lambda(\nu) = (\nu + \frac{1}{2})^2 - (s - \frac{1}{2})^2$.

Corollary

The solution to $(-\Delta)^s u = f$ admits a polyhomogeneous expansion near $\partial\Omega$ with exponents $\nu_j = j + s$.

Extension approach to fractional boundary problems

The analysis of the model problem in the half space implies corresponding results for smooth $\Omega \subseteq \Gamma$, using pseudodifferential techniques.

$$\begin{aligned} (-\Delta)^s u &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{in } \mathbb{R}^n \setminus \overline{\Omega}. \end{aligned} \tag{$\star$}$$

Corollary

The solution to $(-\Delta)^s u = f$ admits a polyhomogeneous expansion near $\partial\Omega$ with exponents $\nu_j = j + s$, i.e. $u \in d^s C^\infty(\overline{\Omega})$, d boundary defining function.

approach generalizes to manifolds with corners, e.g. polygons $\subseteq \mathbb{R}^2$, and lower-order perturbations.

Structure of solution operator

The analysis of the model problem in the half space implies corresponding results for smooth $\Omega \subseteq \Gamma$, using pseudodifferential techniques.

$$\begin{aligned} (-\Delta)^s u = f \quad & \text{in } \Omega, \\ u = 0 \quad & \text{in } \mathbb{R}^n \setminus \overline{\Omega}. \end{aligned} \qquad (\star)$$

Solution operator $\mathcal{S} : f \mapsto u$ has distributional kernel $K(X, Y)$:

$$u(X) = \mathcal{S}f(X) = \int K(X, Y) F(Y)\, dY$$

where $F = L_s(\tilde{f}), \frac{\partial \tilde{f}}{\partial \nu} = f$.

Example: Model problem

$$K(X, Y) = \sum_j B_j \varphi_j^s \otimes \varphi_j^s$$

where

$$B_j(\rho, \tilde{\rho}) = c_j (H(\tilde{\rho} - \rho) I_{j+\frac{1}{2}}(\rho) K_{j+\frac{1}{2}}(\tilde{\rho}) - H(\rho - \tilde{\rho}) K_{j+\frac{1}{2}}(\rho) I_{j+\frac{1}{2}}(\tilde{\rho})).$$

Structure of solution operator

The analysis of the model problem in the half space implies corresponding results for smooth $\Omega \subseteq \Gamma$, using pseudodifferential techniques.

$$\begin{aligned} (-\Delta)^s u &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{in } \mathbb{R}^n \setminus \overline{\Omega}. \end{aligned} \qquad (\star)$$

Theorem (HG, Louca, Mazzeo)

- *Distributional kernel $K(X, Y)$ is C^∞ in $\Omega \times \Omega \setminus \Delta\Omega$.*
- *$K(X, Y)$ lifts to polyhomogeneous distribution on a blown up space $Z^2_{iie} \to \Omega \times \Omega$, i.e. in suitable "polar coordinates" it is a smooth function with classical asymptotic expansions at all boundary faces and product type expansions at corners.*

Nontrivial even for model problem: $K(X, Y) = \sum_j B_j \varphi_j^s \otimes \varphi_j^s$.

The proof of this theorem proceeds with a corresponding fine analysis of the solution operator to the extended problem for L_s in $\mathbb{R}^{n+1}_+$ with mixed boundary conditions.

Conclusions

$$\begin{aligned} (-\Delta)^s u = f \quad & \text{in } \Omega, \\ u = 0 \quad & \text{in } \mathbb{R}^n \setminus \overline{\Omega}. \end{aligned}$$

- Fractional problems interesting & unify FEM/BEM
- Asymptotic expansion of solutions near edges and corners
- Quasi-optimal convergence rates on graded meshes / Exponential convergence *hp*
- Numerical examples confirm analytical expectations

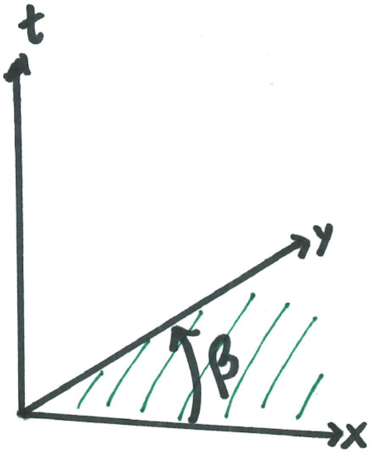
- Construction of solution operator by singular/pseudodifferential analysis of local degenerate elliptic problem in $\mathbb{R}^{n+1}_+$ with mixed boundary conditions

HG, Stephan, Stocek, *hp*-finite element approximation for the fractional Laplacian on uniform and geometrically graded meshes, preprint 2022.

HG, Stephan, Stocek, Corner singularities for the fractional Laplacian and finite element approximation, draft 2022.

Polygonal domains: Model problem

Corner problem: opening angle β



Spherical coordinates

$$t = r\sin(\theta),\ \ x = r\cos(\theta)\cos(\phi),\ \ y = r\cos(\theta)\sin(\phi)$$

$$\widetilde{L}_s U(r,\theta,\varphi) = \partial_r^2 U + \frac{3-2s}{r}\partial_r U + \frac{1}{r^2}\underbrace{\left(\Delta_{\theta,\varphi} U + (1-2s)\tan(\theta)\partial_\theta U\right)}_{=:\widetilde{P}_s U}.$$

Here $\Delta_{\theta,\varphi}$ is the Laplace-Beltrami operator on the 2-sphere S^2:

$$\Delta_{\theta,\varphi} U = \frac{1}{\sin(\theta)}\partial_\theta(\sin(\theta)\partial_\theta U) + \frac{1}{\sin^2(\theta)}\partial_\varphi^2 U.$$

Extension approach for polygons

Spherical coordinates

$$t = r\sin(\theta),\ x = r\cos(\theta)\cos(\phi),\ y = r\cos(\theta)\sin(\phi)$$

$$\widetilde{L}_s U(r,\theta,\varphi) = \partial_r^2 U + \frac{3-2s}{r}\partial_r U + \frac{1}{r^2}\underbrace{\left(\Delta_{\theta,\varphi} U + (1-2s)\tan(\theta)\partial_\theta U\right)}_{=:\widetilde{P}_s U}.$$

Separation of variables:

$\rightsquigarrow$ self-adjoint spectral problem in $L^2(S^2_+, \cos(\theta)^{1-2s} dS)$:

$$\begin{aligned} \widetilde{P}_s \psi = -\lambda\psi \quad & \text{on } S^2_+, \\ \theta^{1-2s} \lim_{\theta\to 0^+} \partial_\theta \psi = 0 \quad & \text{for } \phi \in (0,\beta), \\ \psi = 0 \quad & \text{for } \phi \notin (0,\beta), \theta = \pi/2. \end{aligned}$$

Asymptotic expansion near corner: $u \sim \sum r^{\lambda_j} \log(r)^k u_{jk}(\theta,\phi)$.