Leonardo: (1)

Based on Convex integration for Lip mappings + counterexamples to regularity, by Sverak + Müller

$$I(u) = \int F(\nabla u)\,dx$$

EL: $\mathrm{div}\,(dF(\nabla u)) = 0$

f is rank-1 convex if $t \mapsto f(A+tB)$ is convex $\forall B$ s.t. $\mathrm{rk}(B)=1$.

f is qc if $\int (f(A+\nabla\varphi) - f(A)) \geq 0$.

Ex: det is rk-1 convex and qc

$\mathcal{P}(\mathcal{O}) = \{$ prob. meas. ν s.t. $\mathrm{supp}\,\nu \subseteq \mathcal{O}\}$, $\bar{\nu} = \int x\,d\nu(x)$.

Def: ν is a laminate if $\forall$ rk-1 convex fcts f $\int f(x)\,d\nu(x) \geq f(\bar{\nu})$

Laminates of finite order:

Elementary move: Take δ_A, B_1, B_2 s.t. $\mathrm{rk}(B_2 - B_1) = 1$, $A := sB_1 + (1-s)B_2$, $0 \leq s \leq 1$

$$\delta_A \to \underbrace{\lambda\delta_A + (1-\lambda)(s\,\delta_{B_1} + (1-s)\,\delta_{B_2})}_{= \varphi_{\lambda, A, B_1, B_2}}, \quad 0 \leq \lambda \leq 1.$$

If $\sum \lambda_j \delta_{A_j}$ is laminate $\Rightarrow$ $\varphi_{\lambda, A, B_1, B_2}$ is also laminate.

Def: A laminate of fin. order is ν s.t. it is obtained by $\delta_{\bar{\nu}}$ with finite # of el. moves.

$\mathcal{L}(\mathcal{O}) = \{$laminates of finite order in $\mathcal{O}\}$

Def: Take $K \subseteq \mathrm{Mat}^{m \times n}$ compact. Then K^{rc} is the rk-1 convex hull, $x \notin K^{rc}$ if $\exists f$ rk-1 conv., $f \leq 0$ on K s.t. $f(x) > 0$.
If $\mathcal{O}$ open, $\mathcal{O}^{rc} = \bigcup_{K \subset \mathcal{O}} K^{rc}$, Def.: $\mathcal{P}^{rc} = \mathcal{P} \cap \mathcal{O}^{rc}$

Thm: Let $K \subset M^{r\times n}$ compact, $\nu \in \mathcal{P}^{rc}(K)$, $K^{rc} \subseteq \mathcal{O}$

Then $\exists \nu_j \in \mathcal{Z}(\mathcal{O})$ with $\overline{\nu_j} = \overline{\nu}$ and $\nu_j \xrightarrow{w*} \nu$ in $\mathcal{O}$.

Lemma: $f : \mathcal{O} \to \mathbb{R}$

$$R_{\mathcal{O}} f = \sup \{ g : g \le f : g \text{ is rank-1 convex} \}$$

Then $R_{\mathcal{O}} f = \inf \{ \langle f, \nu \rangle : \nu \in \mathcal{Z}(\mathcal{O}), \overline{\nu} = X \}$

Proof: I: $\inf \ge R_{\mathcal{O}} f$

Take $g \le f$, g rk-1 conv.

$\Rightarrow \langle f, \nu \rangle \ge \langle g, \nu \rangle \ge g(\overline{\nu}) = g(x)$

$\Rightarrow \inf \langle f, \nu \rangle \ge \sup g(x)$.

II: inf is rk-1 convex $\square$

Proof of Thm: $\mathcal{F}$ weak*-closure of $\mathcal{Z}(\mathcal{O})$. Thm $\Leftrightarrow \nu \in \mathcal{F}$.

$\mathcal{Z}(\mathcal{O})$ conv. $\Rightarrow \mathcal{F}$ convex. Suppose $\nu \notin \mathcal{F}$.

Hahn-Banach: $\exists f \in C_0(\mathcal{O})$: $\langle f, \nu \rangle < 0$

$\langle f, \mu \rangle \ge \ \forall \mu \in \mathcal{Z}(\mathcal{O})$

$$\langle f, \nu \rangle \ge \langle R_{\mathcal{O}} f, \nu \rangle \ge R_{\mathcal{O}} f(\overline{\nu}) \underset{\text{Lem}}{=} \inf \{ \langle f, \mu \rangle : \mu \in \mathcal{Z}(\mathcal{O}), \overline{\mu} = \overline{\nu} \} \ge 0$$

↯

Problem: $\nabla u \in K$, $K \subseteq Mat^{m\times n}$, $u : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}^m$

Fine C^0-top.: $\mathcal{U}_{\varepsilon(x)} = \{ v : |v(x)| < \varepsilon(x) \quad \forall x \in \Omega \}$

A fund. nbhd of 0 is $\{ \mathcal{U}_\varepsilon : \varepsilon : \Omega \to (0,\infty) \text{ conti} \}$

Lemma: Let A, B s.t. $rk(A-B) = 1$. Take $C = sA + (1-s)B$,

$u(x) = Cx + b$. Then $\forall \varepsilon$ small enough $\exists v$ Lip, pw. affine

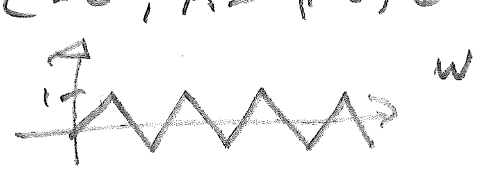
s.t. $\|v-u\|_\infty \le \varepsilon$ and $\nabla v \in \{A, B\}$ a.e.

$|\{\nabla v = A\}| = s\,|\Omega|$.

In Fine C^0: $\exists v_h \to 0$ in fine-C^0: $d(\nabla v, \{A,B\}) < \delta$ $\forall \delta$ small enough

$|\{ d(\nabla v, \{A\}) < \delta \}| = s\,|\Omega|$

Proof: Wlog $C = 0$, $A = (1-s)\, e \otimes e_n^*$, $B = -s\, e \otimes e_n^*$



Take $v_\delta = \delta\, w\left(\frac{x_n}{\delta}\right) e + b$

(Fine C^0: δ depends on region.)

Lemma: $\nu \in \mathcal{L}(M^{m\times n})$, $A = \bar{\nu}$, $\nu = \sum \lambda_j \delta_{A_j}$

$\exists$ pw. affine map v_h s.t. $\nabla v_h \in \{A_j\}_j + B_\delta$ and $v_h \to Ax + b$ in fine C^0

$|\{ \nabla v_h \in \{A_j\} + B_\delta \}| = \lambda_j\, |\Omega|$

Proof: $\square$

Thm: $\mathbb{O}$ open, $P \subseteq \mathbb{O}^{rc}$, u_0 pw. affine Lip s.t. $\nabla u_0 \in P$ a.e.
Then u_0 has a C^0-fine approx. by pw affine Lip maps s.t.
$\nabla u \in \mathbb{O}$ a.e.

Proof: I: One piece of u_0 is enough, $u_0 = Ax+b$ s.t. $A \in K^{rc}_{\Sigma D}$

From previous lectures: $\left\{ \bar{\nu} : \begin{array}{l} \nu \text{ laminate} \\ \nu \in \mathcal{P}(K) \end{array} \right\}$ is dense in K^{rc}

$$\Rightarrow \quad = K^{rc}$$

Take ν s.t. $\bar{\nu} = A$ $\exists \mu_n \in \mathcal{L}(\mathcal{O})$: $\mu_n \to \nu$, $\bar{\mu}_n = A$

Take such a $\mu_n = \sum \lambda_j \delta_{A_j}$, $A_j \in \mathcal{O}$, Apply previous lemma.

Take K. U_j is an in-approx. if $\cdot\ U_i \subseteq U^{rc}_{i+1}$

$\cdot\ \sup_{x \in U_i} d(x, K) \to 0$ as $i \to \infty$.

Thm: Take $K \subseteq M^{m \times n}$ cpct, U_i in-approx. open sets.
If $u_0 : \Omega \to \mathbb{R}^m$ is C^1, $\nabla u_0 \in U_1$, then $\exists$ fine C^0-approx $u_n \to u_0$ and $\nabla u_n \in K$ a.e. in L^p.

Proof: Take u_1 pw. affine s.t. $|u_1 - u_0| \leq \delta_0$, $\nabla u_1 \in U_1$.

Take u_2 s.t. $\nabla u_2 \in U_2$, $|u_2 - u_1| \leq \delta_1$

$\| \nabla u_2 * \rho_{\varepsilon_1} - \nabla u_2 \|_{L^1} \leq 2^{-2}$

$\vdots$

u_n $\nabla u_n \in U_n$ $|u_n - u_{n-1}| \leq \delta_n \leq \varepsilon_n \delta_{n-1}$

$|\nabla u_n * \rho_{\varepsilon_n} - \nabla u_n| \leq 2^{-n}$

$\Rightarrow \nabla u_n \to$ something in L^1.