Nash's embedding reading group

19/01/18

Focus ~~for~~ over next two weeks: "Non-uniquess of weak sol^{n}s to the N-S eq^n", by Buckmaster & Vicol.

For proving existence of sol^{n}s (mild, weak, strong) to N-S by methods of HA, see lectures notes "Harmonic Analysis Tools for solving incompressible NSE" by Cannone.

For the moment, we study NSE on $\mathbb{T}^2$.

> Observe that there is nothing on p in this def; this is because $\Delta p = \operatorname{div}((u\cdot\nabla)u)$

<u>Def</u>: $u \in L^\infty(0,T;\dot H^1(\mathbb{T}^2)) \cap L^2(0,T; L^2_\sigma(\mathbb{T}^2))$ is a <u>weak solution</u> of NSE on $(0,T)$ iff

distrib. $\operatorname{div} u = 0$

$$-\int_0^T\int_{\mathbb{T}^2} \partial_t\phi\cdot u + \nabla\phi\cdot\nabla u + (u\cdot\nabla)\phi \; dx\,dt = 0$$

$$\forall \phi \in C_c^\infty((0,1)^2;\mathbb{R}^2) \quad \nabla\cdot\phi = 0$$

Moreover, we call u a <u>Leray-Hopf weak solution</u> if u additionally satisfies

$$\frac{1}{2}\|u(\cdot,t)\|_2^2 + \int_0^t \|\nabla u(\cdot,t)\|_2^2\,dt \le \frac{1}{2}\|u_0\|_2^2 \quad \text{for a.e. } t\in[0,T].$$

How shall we go about the construction of such an object? For full details, check Robinson's book "Infinite-Dimensional Dynamical Systems" (Ch. 9).

For the moment, we consider Galerkin approximation scheme for

$$\frac{\partial u}{\partial t} + (u\cdot\nabla)u + \nabla p = \Delta u \qquad (*)$$

To find an appropriate system ~~for~~ of finite-dimensional approximants to $(*)$, we begin with

$$u_N(x,t) = \sum_{\substack{|k|\le N \\ k\in\mathbb{Z}^2}} c_k(t)\,\phi_k(x) \qquad x\in\mathbb{T}^2,\ t\in[0,T].$$

Let's consider $\{\phi_k\}_{k\in\mathbb{Z}^2}$ to be the collection of eigenfunctions of the Stokes operator on $\mathbb{T}^2$, i.e.

$$(+_k)\begin{cases} \Delta\phi_k = \lambda_k\phi_k & \text{on } \mathbb{T}^2 \\ \cancel{\nabla\cdot\phi_k = 0}. \end{cases} \qquad \phi_k : \mathbb{T}^2 \to \mathbb{R}$$

Claim: $\{\phi_k\}$ can be considered as an orthonormal basis for $L^2_\sigma(\mathbb{T}^2)$.

Plugging u_N into $(*)$ would yield, amongst other terms,

$$(u_N\cdot\nabla_N)\,u_N:$$

$$\nabla_N u_N^{(t,x)} = \sum_{\substack{|k|\le N \\ k\in\mathbb{Z}^2}} c_k(t)\,\nabla\phi_k(x)$$

$$\Rightarrow \quad u_N\cdot\nabla_N u_N^{(x,t)} = \sum_{\substack{|\ell|\le N \\ \ell\in\mathbb{Z}^2}}\ \sum_{\substack{|k|\le N \\ k\in\mathbb{Z}^2}} c_\ell(t)c_k(t)\,\phi_\ell(x)\,\nabla\phi_k(x).$$

In general, u_N will not be div-free on $\mathbb{T}^2$ $\forall t$. Thus, we consider "Leray projector"

$$\mathbb{P} : L^2(\mathbb{T}^2) \longrightarrow L^2_\sigma(\mathbb{T}^2),$$

i.e. $\mathbb{P}u$ is the divergence-free part of u.

Define also $\mathbb{P}_N : L^2_\sigma(\mathbb{T}^2) \longrightarrow L^2_\sigma(\mathbb{T}^2)$ by

$$\mathbb{P}_N u := \sum_{|k|\le N} (u,\phi_k)\,\phi_k.$$

Instead of studying $(*)$, we shall study, for each N,

$$\frac{\partial u_N}{\partial t} + \mathbb{P}_N\left[(u_N\cdot\nabla)u_N\right] + \nabla p_N = \Delta u_N \quad (*_N).$$

If u_N is chosen as above, then $(*_N)$ reduces to a system of ODEs. In effect, we have a system of ODEs with quadratic nonlinearity.

Prop: There exists $T_* > 0$ such that for any $u_0 \in L^2_\sigma(\mathbb{T}^2)$, u_N is a classical solution of $(*_N)$ on $\mathbb{T}^2 \times (0,1)$ with
$$u_N(\cdot,0) = \mathbb{P}_N u_0 \text{ in } L^2_\sigma(\mathbb{T}^2).$$

Proof: As usual, we hope to demonstrate that $\{u_N\}_{N\geq 1}$ is precompact in an appropriate space. To do so, we appeal to the estimates associated with $(*_N)$.

We begin with
$$\frac{\partial u_N}{\partial t} + \mathbb{P}_N\left[(u_N\cdot\nabla)u_N\right] + \nabla p_N = \Delta u_N$$

$$\Rightarrow \left(\frac{\partial u_N}{\partial t}, u_N\right)_2 + \left(\mathbb{P}_N\left[(u_N\cdot\nabla)u_N\right], u_N\right)_2 + (\nabla p_N, u_N)_2 = (\Delta u_N, u_N)_2$$

$$\Rightarrow \cancel{\tfrac{1}{2}\|u_N(\cdot,t)}$$

u_N divergence-free

$$\Rightarrow \frac{1}{2}\frac{\partial}{\partial t}\|u_N(\cdot,t)\|_2^2 + \Big((u_N\cdot\nabla)u_N, \underbrace{\mathbb{P}_N u_N}_{=u_N}\Big)_2 = -\|\nabla u_N(\cdot,t)\|_2^2$$

$$\Rightarrow \frac{1}{2}\frac{\partial}{\partial t}\|u_N(\cdot,t)\|_2^2 + \|\nabla u_N(\cdot,t)\|_2^2 = -\left((u_N\cdot\nabla)u_N, u_N\right)_2 \quad (\#_N)$$

Need good estimates on RHS in the hope of establishing estimates (indep. on N) on u_N, e.g. $L^\infty(0,T; L^2_\sigma(\mathbb{T}^2))$ and $L^2(0,T;\dot{H}^1(\mathbb{T}^2))$.

We take ~~formally~~ the following from Robinson.

Define
$$b(u,v,w) := \sum_{i,j=1}^{2}\int_{\mathbb{T}^2} u_i \frac{\partial v_j}{\partial x_i} w_j \, dx$$
In particular, the RHS of $(\#_N)$ is given by $-b(u_N,u_N,u_N)$.

Prop (9.2 in Robinson's). For any $v,w,u \in H^1(\mathbb{T}^2)$,
$$|b(u,v,w)| \lesssim \|u\|_2^{1/2}\|u\|_{\dot{H}^1}^{1/2}\|v\|_{\dot{H}^1}\|w\|_2^{1/2}\|w\|_{\dot{H}^1}^{1/2}.$$

~~From (H_N) and noting that $b(u_N, u_N, u_N) = 0$:~~

In (H_N), note that $b(u_N, u_N, u_N) = 0$:

$$b(u,u,u) = \int_{\mathbb{T}^2} u_i \frac{\partial u_k}{\partial x_i} u_k \, dx =$$

$$= \cancel{\frac{1}{2}\int_{\mathbb{T}^2} u_i \frac{\partial}{\partial}} = \frac{1}{2}\sum_k \int_{\mathbb{T}^2} u_i \frac{\partial}{\partial x_i}\left[(u_k)^2\right] dx$$

$$= -\frac{1}{2}\sum_k \int_{\mathbb{T}^2} \nabla\cdot u \left[(u_k)^2\right] dx \overset{\nabla\cdot u = 0}{=} 0$$

Then, in (H_N) we have

$$\frac{1}{2}\frac{\partial}{\partial t}\|u_N(\cdot,t)\|_2^2 + \|\nabla u_N(\cdot,t)\|_2^2 = 0, \text{ then } \int_0^t d\tau,$$

$$\frac{1}{2}\|u_N(\cdot,t)\|_2^2 + \int_0^t \|\nabla u_N(\cdot,\tau)\|_2^2 \, d\tau = \frac{1}{2}\|\mathbb{P}_N u_0\|_2^2$$
$$\leq \frac{1}{2}\|u_0\|_2^2 \quad \forall t \in (0,T^*).$$

Take sup:

$$\frac{1}{2}\sup_{0\leq t\leq T^*}\|u_N(\cdot,t)\|_2^2 + \int_0^{T^*}\|\nabla u_N(\cdot,t)\|_2^2 \, dt \leq \frac{1}{2}\|u_0\|_2^2$$

so $u_N \in L^\infty(0,T^*; L^2_\sigma(\mathbb{T}^2)) \cap L^2(0,T^*; \dot{H}^1(\mathbb{T}^2))$, and by boundedness of the seq. u_N in these norms we have existence of a weak limit in these spaces.

We investigate what troubles we may have in "passing to the limit" in $(*_N)$ as $N\to\infty$.

If $(*_N)$ holds classically

$$-\int_0^{T^*}\int_{\mathbb{T}^2} u_N(x,t)\cdot\partial_t\phi(x,t) - \int_0^{T^*} b(u_N, \mathbb{P}_N\phi, u_N)(x,t)$$
$$+\int_0^{T^*}\int_{\mathbb{T}^2}\nabla u_N(x,t)\cdot\nabla\phi(x,t) = 0$$

The first and the third terms converge to what is needed. The only troubles could be given by the term with the trilinear form.

$$b(u_N, u_N, u_N) - b(u,u,u) =$$

$$= \underbrace{\int_0^{T^*}\int_{\mathbb{T}^2} (u_n - u)\cdot \nabla \mathbb{P}_N \phi \; u_N}_{I_1} + \int_0^{T^*}\int_{\mathbb{T}^2} (u\cdot\nabla)\mathbb{P}_N u_N - (u\cdot\nabla)\phi u$$

$$= I_1 + \int_0^{T'}\int_{\mathbb{T}^2} (u\cdot\nabla)(\mathbb{P}_N - \mathbb{P})\phi \, u_N + (u\cdot\nabla)\,\mathbb{P}\phi \cdot u_N - (u\cdot\nabla)\,\mathbb{P}\phi u.$$

All this stuff weakly converges to 0. We have weak convergence in time and space.

Aubin-Lions lemma will allow you to improve convergence in time.