Mark: Lecture 1

What are we going to do? We are going to use Nash's ideas to demonstrate non-uniqueness of weak solutions to PDE in fluid mechanics and continuum mechanics, more generally.

In particular, we shall use Nash's ideas to understand the status (in modern PDE theory) of Hadamard's well-posedness:

Hadamard well-posedness for an IVP:

$$u_t = F(u, Du, D^2u)$$
$$u|_{t=0} = u_0 \in X \quad \text{functional class}$$

(H1) existence

(H2) uniqueness

(H3) continuous dep. on initial data.

Also of interest will be the interpretation of nonuniqueness from the viewpoint of continuum mechanics.

Which equations are we going to study?

(i) incompressible Euler on $\mathbb{T}^3$: $\begin{cases} \partial_t u + (u\cdot\nabla)u + \nabla p = 0, \quad \nabla\cdot u = 0 \\ u|_{t=0} = u_0 \end{cases}$

De Lellis + Székelyhidi Jr (The h-principle and the equations of fluid mechanics)

(ii) Incompressible Navier-Stokes on $\mathbb{T}^3$:

$$(*) \begin{cases} \partial_t u + (u\cdot\nabla)u + \nabla p = \nu\Delta u \\ u|_{t=0} = u_0, \quad \nabla\cdot u = 0 \end{cases}$$

Buckmaster + Vicol (Non-uniqueness of weak solutions to the Navier-Stokes equations).

Def: We shall say that $u \in C^0(\mathbb{R}; L^2(\mathbb{T}^3))$ is a weak solution of $(*)$ iff

- u is div. free in $\mathcal{D}'$
- u has zero mean over $\mathbb{T}^3$

and $\quad \cdot \int_{\mathbb{R}} \int_{\mathbb{T}^3} u \cdot (\partial_t \varphi + (u\cdot\nabla)\varphi + \nu \Delta \varphi)\, dx\, dt = 0 \quad \forall \varphi \in C_0^\infty(\mathbb{T}^3 \times \mathbb{R})$ div. free

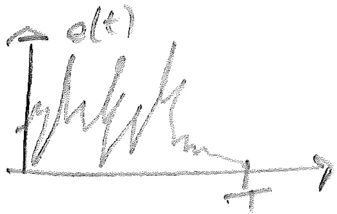
N.B.: Leray-Hopf weak solutions also involve the energy inequality for NS.

The reason one does not have an energy identity in the paper of B-V is that they do not treat the IVP for NS.

Thm 1.2 [BV]: There exists $\beta > 0$ s.t. $\forall$ smooth, non-negative $e: [0,T] \to \mathbb{R}_+$, there exists $u \in C^0([0,T], H^\beta(\mathbb{T}^3))$ which is a weak solution of NS and

$$\int_{\mathbb{T}^3} |u(x,t)|^2 dx = e(t) \quad \forall t \in [0,T].$$

For example:



This is clearly a non-uniqueness result for the problem

$$\begin{cases} \partial_t u + (u\cdot\nabla) u + \nabla p = \nu \Delta u \\ u|_{t=0} = 0, \quad \nabla \cdot u = 0 \end{cases}$$

It does not, however, yield a non-uniqueness result for the IVP with $u|_{t=0} = u_0 \in L^2_\sigma(\mathbb{T}^3) \setminus \{0\}$.

The state of the art is better for the incompressible Euler equations.

See Daneri + Szekelyhidi Jr. (Non-uniqueness and h-principle for Hölder continuous weak solutions of the Euler equation).

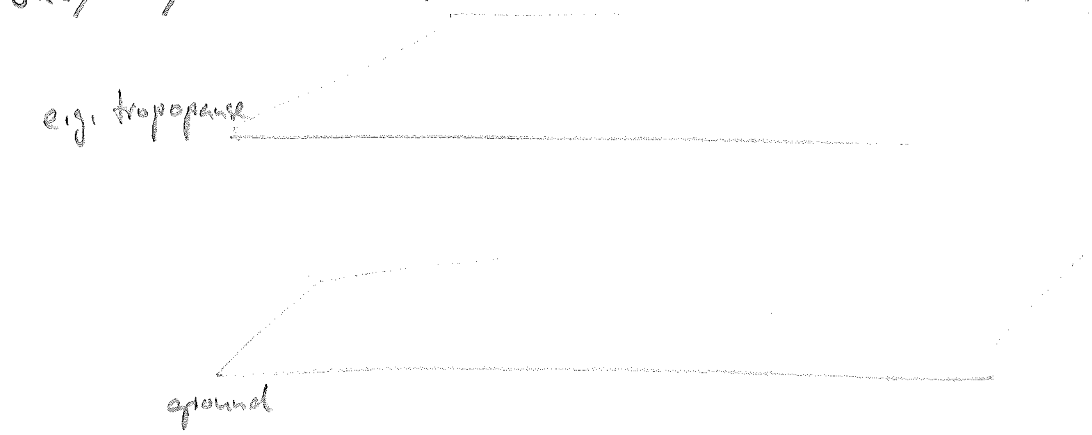
(iii) The surface quasi-geostrophic equation

geostrophic balance from atmospheric science
coriolis force $\sim \nabla p$, p pressure

$$\partial_t \vartheta + (u\cdot\nabla)\vartheta = 0 \quad \text{on } \mathbb{T}^2$$
$$u = \nabla^\perp (-\Delta)^{1/2} \vartheta, \quad \vartheta|_{t=0} = \vartheta_0$$

This is an example of an active scalar equ. See also the work of

Shvydkoy (convex integration for a class of active scalar eqns.)



(iv) <u>Transport equations</u>

$$(**)\begin{cases} \partial_t \rho + (u\cdot\nabla)\rho = 0 \\ \rho|_{t=0} = \rho_0 \;, \quad u : [0,T)\times \mathbb{T}^d \to \mathbb{R}^d \text{ given}, \end{cases}$$

For existence of solutions when u is Sobolev : DiPerna + Lions (Annals '89). For $u \in BV$: Ambrosio (Inventiones ?)

It has been demonstrated by Modena + Székelyhidi Jr that "solutions" of $(**)$ where u is Sobolev are non-unique.

What is <u>convex integration</u>?

It is a technique / construction scheme one employs which at each stage of the construction contains great deal of of arbitrariness as to how to progress to the next stage.

To fix ideas to how things "go wrong" (from the point of view of uniqueness), let us consider the incompressible NS equations on $\mathbb{T}^d$ $(d=2,3)$.

Indeed,
$$(+)\begin{cases} \partial_t u + (u\cdot\nabla)u + \nabla p = \Delta u \\ u|_{t=0} = u_0 \in L^2(\mathbb{T}^d) \end{cases}$$

One way by which to construct global in time weak solutions to $(+)$ is to use a Galerkin scheme:

$$(+_n)\begin{cases} \partial_t u_n + \mathbb{P}_n[(u_n\cdot\nabla)u_n] + \nabla p_n = \Delta u_n \\ u|_{t=0} = \mathbb{P}_n u_0 \in L^2(\mathbb{T}^d) \end{cases}$$

where $\mathbb{P}_n$ is the Leray projector onto divergence-free maps in $L^2_\sigma(\mathbb{T}^d)$
$= \{v \in L^2(\mathbb{T}^d, \mathbb{R}^d) : \nabla \cdot v = 0 \text{ in } \mathcal{D}'(\mathbb{R}^d)\}$

e.g. $u_n(x,t) = \sum_{k=1}^{n} c_k^{(n)}(t)\, \phi_k(x)$, $\{\phi_k\}_k$ ONB of eigenfunctions to the Stokes operator on $\mathbb{T}^d$.

For uniqueness of weak solutions of (+), let's look at $d=2$ first.
Suppose we have 2 solutions of (+), $u, v \in L^2(0,T, H^1(\mathbb{T}^2)) \cap L^\infty(0,T, L^2(\mathbb{T}^2)) \cap C^0([0,T], L^2(\mathbb{T}^2))$. Note that (formally)

$$\partial_t (u-v) + ((u-v)\cdot\nabla)u + (v\cdot\nabla)(u-v) + \nabla p_1 - \nabla p_2 = \Delta u - \Delta v.$$

$$\Rightarrow \int_{\mathbb{T}^2} \left\{ \partial_t (u-v)(u-v) + ((u-v)\cdot\nabla)u\cdot(u-v) + (v\cdot\nabla)(u-v)\cdot(u-v) \right\} dx$$

$$= - \int_{\mathbb{T}^2} |\nabla u - \nabla v|^2\, dx$$

$$\Rightarrow \frac{1}{2}\partial_t \int_{\mathbb{T}^2} |u-v|^2\, dx + \int_{\mathbb{T}^2} ((u-v)\cdot\nabla)u\cdot(u-v)\, dx$$

$$+ \int_{\mathbb{T}^2} |\nabla u - \nabla v|^2\, dx = 0$$

We need to estimate the "messy" terms. E.g.

$$\int_{\mathbb{T}^2} ((u-v)\cdot\nabla)u\cdot(u-v)\, dx$$

(Hölder) $\leq \|u(\cdot,t) - v(\cdot,t)\|_4^2 \, \|\nabla u(\cdot,t)\|_2$

We use Ladyzhenskaya's inequality in $d=2$: $\|u\|_{L^4} \leq C \|u\|_{L^2}^{1/2} \|\nabla u\|_{L^2}^{1/2}$

$\leq \|u-v\|_{L^2} \|\nabla u\|_{L^2} \|\nabla(u-v)\|_{L^2}$

$\leq \frac{C}{\varepsilon} \|u-v\|_{L^2}^2 \|\nabla u\|_{L^2}^2 + \varepsilon \|\nabla u - \nabla v\|_{L^2}^2$

Using this, we obtain

$$\frac{d}{dt} \|u(\cdot,t) - v(\cdot,t)\|_{L^2}^2 + \|\nabla u - \nabla v\|_2^2$$

$$\leq \frac{C}{\varepsilon} \|u - v\|_{L^2}^2 \underbrace{\|\nabla u\|_{L^2}^2}_{\nabla u \in L_t^2 L_x^2} + \varepsilon \|\nabla u - \nabla v\|_{L^2}^2$$

$$\Longrightarrow \quad \frac{d}{dt} \|u - v\|_{L^2}^2 \leq \frac{C}{\varepsilon} \|u - v\|_{L^2}^2 \|\nabla u\|_{L^2}^2$$

By applying Gronwall's ineq.

$$\|u - v\|_{L^2}^2 \leq C(t) \|u_0 - v_0\|_{L^2}^2$$

Another claim: This won't work in 3d due to "poor scaling" properties of Ladyzhenskaya.