Coming lectures: Nonsmooth weak solns. to physical PDEs: cont. mech. Nash R.G.

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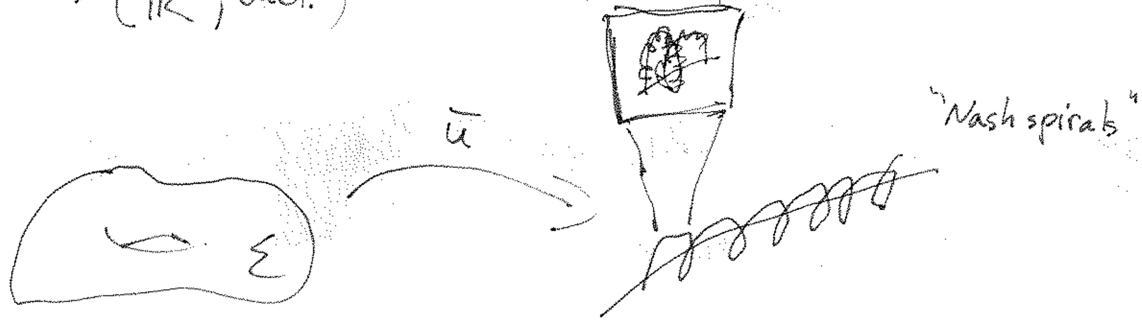
Calculus of variations (Šverak & Müller)

Starting point: Nash's nonsmooth embedding thm.

(Σ, g) Riem. mfld., $g \in C^\infty$ $\Rightarrow$ $\exists N = N(n)$ and an isometric C^1-embedding

$(\Sigma, g) \xrightarrow{u} (\mathbb{R}^N, \text{eucl.})$ (1)

Idea:



Idea of proof: Adding oscillations to a strictly short embedding gives rise to better approx. of metric. Iterate. Convergence of u_j to isometry u in C^1 not in C^2.

1) Thm: $\forall \varepsilon > 0$ $\forall u : (\Sigma, g) \longrightarrow (\mathbb{R}^N, \text{eucl.})$ short, C^∞ $\Rightarrow \exists v : \|u - v\|_{C^0} < \varepsilon$ and v isometric embedding.

Threshold results for embedding problems:

C^1: isometric embeddings are C^0-dense in space of all short embeddings

C^2: rigidity results: image of a C^2-embedding is highly constrained:

The image of an isometric $C^{1,\theta}$-embedding (or just immersion) $: S^2 \longrightarrow \mathbb{R}^3$ is the boundary of an open convex set.

This is result by Borisov for $C^{1,\theta}$, for C^k, k large: Weyl.

Threshold phenomenon: density of $C^{1,\theta}$ isom. embeddings for $\theta < \theta_0$, constr. for $\theta > \theta_0$.

Case θ_0 very difficult: de Lellis et al: $\theta_0 > \frac{1}{1+n(n+1)^2}$

when Σ is a ball: $\theta_0 > \frac{1}{1+n(n+1)}$

θ_0 depends on codimension; In high codimension $(\Sigma, g) \longrightarrow \mathbb{R}^N$, N 'huge'

We can get "almost" to C^2, $\theta_0 \approx 1$.

<u>Källen:</u> (Σ, g) Riem. mfld., $g \in C^{1,\theta} \Rightarrow$ ~~short~~ $\exists u: (\Sigma, g) \longrightarrow (\mathbb{R}^N, \text{eucl.})$ isometric s.t. $\|u - v\|_{C^0} < \varepsilon$.

$\exists N; \forall v: (\Sigma, g) \to (\mathbb{R}^N, \text{eucl})$ short, $\forall \varepsilon > 0$

<u>Euler equations (and what Mark will talk about)</u>

$$\partial_t \vec{v} + (\vec{v} \cdot \nabla)\vec{v} + \nabla p = 0 \qquad ; \quad \vec{v}: \mathbb{R}^+_t \times \mathbb{R}^3_x \longrightarrow \mathbb{R}^3$$
$$\operatorname{div} \vec{v} = 0 \qquad \text{or } \mathbb{R}^+_t \times \mathbb{T}^3_x \longrightarrow \mathbb{R}^3$$
$$p: \mathbb{R}^3 \longrightarrow \mathbb{R}$$

Classical solns. are just $C^1(\mathbb{R}^+ \times \mathbb{T}^3)$ + satisfy eqns.

<u>Problem:</u> Little is known. Physics gives rise to more general solns.; such as sharp wave crests, turbulence, etc.

(i) Local well-posedness: $\exists!$ soln. in $C^{1,\theta}([0,T] \times \mathbb{T}^3)$ for T small enough. (also H^s $s > \frac{5}{2}$)

(ii) Beale-Kato-Majda: If $\int_0^T \|\nabla \times \mathbf{v}\|_{L^\infty} < \infty$, then soln. extends to $[0, T+\varepsilon]$.

Weak Solutions: Existence ✓.

(different notions)

Thm: [Sheffer '93] There exist infinitely many nontrivial weak solns in $L^\infty(\mathbb{R}\times\mathbb{T}^3)$ with compact support in time.

How constrained are weak solns. to the Euler equations?

Onsager conjecture / Isett's thm (2016)

Energy $E(t) = \frac{1}{2}\int_{\mathbb{T}^3} |\vec{v}(t,x)|^2 dx$, compare "metric" in Nash's work.

Exercise: If $v \in C^1$, then $\frac{d}{dt}E(t) = 0$.

For Sheffer's solns: $E(t) \not\equiv$ const.

Questions:

- Which functions $E(t)$ are energies of solutions to the Euler equations?
- Under which conditions on v is $E(t) =$ const.?

Answer: If $v \in C^{\frac{1}{3}+\varepsilon} \Rightarrow$ ——

If $v \in C^{\frac{1}{3}-\varepsilon}$: any $E(t)$.