C^1 isometric embedding.

Let (Σ, g) Riemannian manifold, $v: \Sigma \to \mathbb{R}^N$ short if $v^\# e_{\text{ucl.}} \leq g$.

- $v^\# e_{\text{ucl.}} = (\partial_i v) \cdot (\partial_j v) dx_i \otimes dx_j$
- $h \leq g \iff h_{ij} \xi_i \xi_j \leq g_{ij} \xi_i \xi_j \quad , \forall i,j$.

deLellis: • Masterpieces of Nash
• Nash's nonlinear iteration

Thm: (Nash-Kuiper)
- (Σ, g) smooth, closed, $\dim \Sigma = n$.
- $v: \Sigma \to \mathbb{R}^N$, C^∞ short, $N \geq n+2$ (immersion)

$\Rightarrow \forall \varepsilon > 0 \; \exists$ isometric C^1-immersion $u: \Sigma \to \mathbb{R}^N$; $\|u - v\|_{C^0} < \varepsilon$.

If v is an embedding, then we can choose u to be an embedding.

Corollary: Any ~~...~~ smooth n-dim Riemannian manifold has a C^1-isometric immersion in $\mathbb{R}^{2n}$ and a C^1-isometric embedding in $\mathbb{R}^{2n+1}$. If the manifold is closed, then $\exists$ C^1-isometric embedding in $\mathbb{R}^{2n}$.

Note: While the statement of the thm. is geometric, the key ingredients of the proof are PDE techniques. Like for the smooth embedding thm., the key idea is a clever nonlinear iteration.

In order to get from geometric to PDE problem in $\mathbb{R}^n$, we require a suitable choice of local coords. The choice as in the smooth case does the job.

Key proposition (2.1. deLellis, NNI):

(Σ, g) as in Thm., $z: \Sigma \to \mathbb{R}^N$, C^∞ strictly short immersion

Then, $\forall \eta, \delta > 0$, $\exists$ smooth short $z_1: \Sigma \to \mathbb{R}^N$, s.t. $\|z - z_1\|_0 \leq \eta$

$\|g - z_1^\# e_{\text{ucl.}}\|_0 \leq \delta$, $\|Dz_1 - Dz\|_0 \leq C \sqrt{\|g - z^\# e_{\text{ucl.}}\|_0}$, $C = C(\Sigma)$. If z is injective, then z_1 can be injective.

Proof (of Thm. using Prop.): (only for immersions) ①

Let $v =: v_0$, WLOG v strictly short. Apply prop. with $z = v_0$ and $\eta = \frac{\varepsilon}{4}$, $\delta = \frac{1}{4}$. Get $z_1 =: v_1$.

Iterate the prop. with $z = v_j$, $\eta = \frac{\varepsilon}{2^{j+2}}$, $\delta = \frac{1}{4^{j+1}}$ $\rightsquigarrow z_1 =: v_{j+1}$.

From triangle ineq. $\rightsquigarrow \{v_j\}$ is Cauchy in C^0, with limit u

$$u = \lim_{j\to\infty} \left(v_0 + (v_1 - v_0) + (v_2 - v_1) + \dots\right) \text{ exists,}$$

$\|v_0 - u\|_0 \le \frac{\varepsilon}{2}$. Also, $\|Dv_{j+1} - Dv_j\|_0 \le C\sqrt{\|g - v_j^\# \mathrm{eucl}\|_0} \le C\sqrt{\delta_j}$

Thus convergence in C^1, not just C^0.

Thus $v^\# \mathrm{eucl.} \xrightarrow{C^0} u^\# \mathrm{eucl.}$ Thus u is immersion.

- Check that u is **injective**. (for embeddings)

Analyst's key prop.:

Define $\mathcal{C}_r := \left\{ A \in \mathrm{Sym}_{n\times n} : \left\| \frac{A}{\frac{1}{n}\mathrm{tr} A} - \mathbb{1} \right\| < r \right\}$. cone of symmetric matrices in neighborhood of $\mathbb{1}$.

Key prop: $\exists\, r_0 = r_0(n)$ s.t.: If $U \subseteq \mathbb{R}^n$ is bounded simply connected open domain, $g \in C^\infty(\overline{U})$ smooth metric $z \in C^\infty(\overline{U} \to \mathbb{R}^N)$ smooth short map with $g - z^\# \mathrm{eucl} \in \mathcal{C}_{r_0}$ $\forall x \in \overline{U}$, then $\forall \delta, \eta > 0$, $\exists$ smooth short map $z_1 : \overline{U} \to \mathbb{R}^N$
$g - z_1^\# \mathrm{eucl} \in \mathcal{C}_{r_0}$, $\forall x \in \overline{U}$ and

$$\|z - z_1\|_0 \le \eta, \quad \|g - z_1^\# \mathrm{eucl}\|_0 \le \delta, \quad \|Dz - Dz_1\|_0 \le M_n \sqrt{\|g - z_1^\# \mathrm{eucl}\|_0}$$

If z injective then z_1 can be injective.

We decompose the metric into "primitive" matrics.

Lemma: (2.4. NNI) Let $S_+^{n\times n}$ be the set of pos. def. symmetric matrices. Then $\exists$ neighborhood W of $\mathbb{1}$ and $N(n) = \frac{n(n+1)}{2}$ unit vectors $v_k \in \mathbb{R}^n$: any $A \in W$ can be written as

$$A = \sum_{k=1}^{N(n)} \lambda_k(A)\, v_k \otimes v_k \quad : \lambda_k(A) \text{ linear in } A \text{ and}$$

$$\lambda_k(A) \geq \rho_0(n) \text{ for some } \rho_0(n) > 0.$$

Proof of analyst's key prop.: $z^\# eucl = Dz^T Dz$

By assumption: $g - z^\# eucl = g - Dz^T Dz \in C_{r_0}$ which is positive def.

$\rightsquigarrow g - Dz^T Dz \geq 2\gamma \mathbb{1}$ for some $\gamma > 0$. WLOG $\gamma \leq \delta$.

Let $h(x) = g(x) - Dz^T Dz(x) - 2\gamma\mathbb{1} \in C_{r_0}$.

By Lemma: $h(x) = \sum_{k=1}^{N(n)} a_k^2(x)\, v_k \otimes v_k =: \sum_k h_k$
C^∞ as lin. in h.

(1) $z \to \bar{z}$: $D\bar{z}^T D\bar{z} \simeq Dz^T Dz + h_1$

Prop: U as above $a_1 \in C^\infty(\bar{U})$, v_1 unit vector $z \in C^\infty(\bar{U} \to \mathbb{R}^N)$ short. Then $\forall \tilde{\delta}, \tilde{\eta}$, $\exists\, \bar{z} \in C^\infty(\bar{U} \to \mathbb{R}^N)$ short s.t.

$$\|\bar{z} - z\|_0 < \tilde{\eta}, \quad \|D\bar{z}^T D\bar{z} - Dz^T Dz - a_1^2 v_1 \otimes v_1\|_0 \leq \tilde{\delta}$$
$$\|D\bar{z} - Dz\|_0 \leq \tilde{M}_n \|a_1\|_0$$

Pf.: $\bar{z}(x) = z(x) + z_p(x)$, where $z_p(x)$ defined as follows:

Choose unit length normal fields, $v, b : \bar{U} \to \mathbb{R}^N$: $v(x) \perp b(x)$, $\forall x$.
$v(x), b(x) \perp T(z(U))$.

$$z_p = \frac{a_1(x)}{\lambda}\left(v(x)\cos(\lambda \vartheta_1 \cdot x) + b(x)\sin(\lambda \vartheta_1 \cdot x)\right).$$

λ - to be chosen to make the estimates work ($\lambda \gg 1$) [Nash Spirals].

- First estimate is clear.
- Third estimate follows directly from differentiation of sin/cos and cancel $\frac{1}{\lambda}$.
- Second estimate: Compute $D\bar{z}^T D\bar{z} - Dz^T Dz$.

$$D\bar{z} = Dz + Dz_p$$

$$D\bar{z}^T D\bar{z} = Dz^T Dz + \underbrace{Dz^T Dz_p + Dz_p^T Dz}_{=:L} + \underbrace{Dz_p^T Dz_p}_{=:Q}$$

$$Dz_p(x) = \underbrace{-a_1(x)\sin(\lambda\vartheta_1\cdot x)\, v(x)\otimes\vartheta_1}_{=:A} + \underbrace{a_1(x)\cos(\lambda\vartheta_1\cdot x)\, b(x)\otimes\vartheta_1}_{=:B}$$
$$+ \text{l.o.t.}_\lambda \to 0 \text{ as } \lambda\to\infty$$

The key observation is that Q is dominant.

$$A^T Dz = 0 = B^T Dz \quad \text{by orthogonality}$$
$$Dz^T A = 0 = Dz^T B$$

$$\rightsquigarrow \|L\|_0 \le C\|\text{error}\| \to 0$$

But $\begin{pmatrix} D\bar{z}^T D\bar{z} \\ -Dz^T Dz\end{pmatrix} \approx (A+B)^T(A+B) = a_1^2\,\vartheta_1\otimes\vartheta_1 + \underbrace{A^T B + B^T A}_{=0}$

$$\rightsquigarrow \|D\bar{z}^T D\bar{z} - Dz^T Dz - a_1^2\,\vartheta_1\otimes\vartheta_1\|_0 \le \delta$$

$\square$