Nash 2 group Heiko

Recall:

Dini's thm: F, G Banach. $P: U \subseteq F \to V \subseteq G$ smooth, U, V open. Suppose that that for $f_0 \in U$, $DP(f_0)$ is invertible. Then $\exists\ \tilde{U} \subseteq U$ open, $f_0 \in \tilde{U}$, s.t. $\tilde{V} := P(\tilde{U})$ is open and P is bijective, and $P^{-1}: \tilde{V} \to \tilde{U}$ is continuous.

Rk: In this case, P^{-1} is smooth.

Rk: Invertible linear maps are not open in Fréchet spaces.

Ex: $(L(t)f)(x) = \frac{1}{t}\int_x^{x+t} f(u)\,du$, $L: \mathbb{R} \times C^\infty_{2\pi} \to C^\infty_{2\pi}$

$L(0)f = f \Rightarrow L(0) = Id$, but $L\left(\frac{2\pi}{n}\right)\sin(nx) = 0$, so L not one-to-one in a neigh. of $t=0$.

Rk: This is not a counterexample in $C^0_{2\pi}$ because $L(t) \not\to L(0)$ in op. norm on $C^0_{2\pi}$ (need opnorm for continuity

Ex. 2: Surj can fail too $P: C^\infty(-1,1) \to C^\infty(-1,1)$

$Pf := f - xf\frac{df}{dx}$, $DP(f)[g] = g - xgf' - xfg'$.
$DP(0) = Id$, but

But $g_n = \frac{1}{n} + \frac{x^n}{n!}$ is not in $R(P)$;

$DP(1/n)\, x^k = (1 - k/n)\, x^k$, so

$x^n \notin R\left(DP(1/n)\right)$.

Thm for an impl. f. thm, we need the assumption

$DP(f)$ inv. in a neighborhood

Rk: (Worse pb.) We will need restrictions on the class of ~~spaces~~ operators.

$E^n := \{ f \in C(\mathbb{R}) \mid f(x) = 0\ \forall x \leq 1,\ \underbrace{\sup_x e^{-nx} |f(x)|}_{\|f\|_n} < +\infty \}$

$E^\infty := \bigcap_{n=1}^{+\infty} E_n$ Fréchet. Take $Lf(x) := f(x/2)$.

Then $L : E^{2n} \to E^n$ isometry $\Rightarrow L : E^\infty \to E^\infty$ inv.

Take $Pf(x) := f(x) - (Lf(x))^2$. Then $P : E^n \to E^n$ continuous,

Thm: P is inv. near 0 on E^n: if $g \in E^n$ and

$\|g\|_n < 1/4$, then $\exists! f$ s.t. $P(f) = g$ and $\|f\| < 1/2$.

Pf: $g(x) = f(x) - \left(f(x/2)\right)^2$, or $f(x) = g(x) + f(x/2)^2$.

Iterate $\rightsquigarrow$ you converge.

Thm: P is not invertible near 0, but DP is.

Pf: Define $f_m \in E^m$, $f_m(x) = \begin{cases} 0 & x \leq 1 \\ (x-1)e^{-2m} & 1 \leq x \leq 2 \\ e^{-mx} & x \geq 2 \end{cases}$.

Then $P(f_m) = 0 \quad \forall x \geq 4 \Rightarrow P(f_m) \in E^\infty$.

$\|f_m\|_n = e^{2(n-m)}$. (for $m \geq m_0(n)$).

Therefore $f_m \to 0$ in $E^\infty \overset{\text{continuity}}{\Rightarrow} P(f_m) \to 0$ in E^∞

By the pr. thm., $P^{-1}(\underbrace{P(f_m)}_{E^\infty}) = f_m \notin E^\infty$.

Therefore, $P(f_m) \notin R(P|_{E^\infty})$.

<u>Tame Fréchet spaces:</u>

Let $(F, \|\cdot\|_n)$ Fréchet space

<u>Def</u>: $(F, \|\cdot\|)$ is <u>graded</u> if $\|f\|_j \geq \|f\|_k \quad \forall j \geq k \quad \forall f \in F$

<u>Rk</u>: By defining $\|\cdot\|'_n = \sum_{k \leq n} \|\cdot\|_k$, every Fréchet norm is graded

<u>Def</u>: Let F, G graded. Then $L: F \to G$ is <u>tame</u> of deg $= r$ and base $= b$ iff $|||Lf|||_n \leq C_n \|f\|_{n+r} \; \forall n \geq b$

<u>Rk</u>: Comp. of Tame is Tame (with different base and + of degrees.

<u>Rk</u>: Any linear diff. op. is Tame in C^∞.

<u>Rk</u>: Tame $\Rightarrow$ cont.

Def: L is a tame isomorphism if L is a linear isomorphism and both L, L^{-1} are tame.

Def: $P: F \to G$, F, G graded, $P: F \to G$ nonlinear, P is smooth tame if P smooth and $D^k P$ is tame $\forall k$.

Thm: (Nash-Moser)

If F, G are tame Fréchet, $P: U \subseteq F \to V \subseteq G$ smooth tame, suppose $DP(f)[h] = k$ admits a unique sol. $h = VP(f)k$ $\forall f \in U$ $\forall k$, and $VP: U \times G \to F$ is smooth tame. Then P is locally invertible and each local inverse P^{-1} is smooth tame.

Pf: (Sketch)

Surj: WTS $Pf = g$. In Banach, solving $\begin{cases} \frac{d}{dt} f_t = c\, VP(f_t)[g - P(f_t)] \\ f_0 = 0 \end{cases}$ works (take the lim as $t \to \infty$):

$$DP(f_t) \frac{d}{dt} f_t = c\,[g - P(f_t)] \Rightarrow$$

$$\frac{d}{dt} P(f_t) = c\,[\underbrace{g - P(f_t)}_{=k_t}] \Rightarrow \frac{d}{dt} k_t + ck = 0$$

$$\Rightarrow k_t = e^{-ct} g \to 0$$

In the Fréchet setting, we want to regularise:

$$\frac{d}{dt} f_t = c\, \nabla P\left(S_{1/t} f_t\right) S_{1/t}\left[g - P(f_t)\right].$$