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Smooth isometric embeddings & inverse fc. theorems in Frechét spaces

~~Abstract~~: Embed (Σ, g) $\nearrow$ C^k isometric embeddings into $\mathbb{R}^N$, $k \geq 3$ (Nash-Moser-Hörmander iterations) IFT's in Frechét spaces

(Σ, g) $\searrow$ C^α isometric embeddings $\alpha = 1$. Later on. (Convex integration/ h-principle) [C. DeLellis ...?]

[R.S. Hamilton - The inv. funct. thm. of Nash & Moser, BAMS '82]

[L. Hörmander, BVP ... physical ARMA '78]

[C. DeLellis, The masterpieces of J.F. Nash, homepage]

Thm. (Nash) (Σ, g) closed n-dim. Riemannian manifold with metric of class C^k, $k \geq 3$, $n \geq 1$, $N = \frac{n(3n+11)}{2}$
$\Rightarrow \exists$ isometric embedding $u: \Sigma \to \mathbb{R}^N \in C^k$.

Recall: Manifold has local coords $(x_1, \ldots, x_n)$. Assume local patches are "nice" (balls). Only finitely many patches overlap in any given point.

Then, $g = \sum_{i,j} g_{ij}\, dx_i \otimes dx_j$ is metric in local coords.
g_{ij}: These are C^k.

If T is any tensor field on Σ: similarly define C^k-norms

$$\|T\|_{C^k} = \sup_{\substack{\text{coord.}\\ \text{patches}}} \|T_{(i_1, \ldots, i_k)}\|_{C^k(\mathbb{R}^N)}$$

Why is Thm. a PDE Question?

Thm. $\Leftrightarrow$ Find $u: \Sigma \to \mathbb{R}^N$ embedding s.t. $(\partial_i u) \cdot (\partial_j u) = g_{ij}$, $\forall i,j$.

invariant way of writing is: $u^\# \mathrm{eucl}_{\mathbb{R}^N} = g$

$$u^\# \mathrm{eucl}_{\mathbb{R}^N} = \sum (\partial_i u) \cdot (\partial_j u)\, dx_i \otimes dx_j.$$

Remark: N has been lowered later by Gromov, Günther $\rightsquigarrow N = 2n$.

(2) <u>Remark:</u> Günther showed that if you work in $C^{k,\alpha}$ spaces, $\alpha > 0$, then standard Banach fixed point arguments suffice.

- But techniques were found to be useful: KAM theory of perturbs of integrable systems (Moser), inverse/implicit function thms in Frec sp

<u>Strategy:</u> Linearise + Perturbation result $\equiv$ implicit function theorem

Start with $W_0 = (w_1, \ldots, w_N)$ smooth embedding $\Sigma \to \mathbb{R}^N$.

Take $h = g - W_0^{\#}$ eucl. (Ideally $h = 0$)

- If h is small, find $u : \Sigma \to \mathbb{R}^N$ smooth embedding

find $u^{\#}$ eucl $- w_0^{\#}$ eucl. $= h$. If we can find $u \Rightarrow$ Result

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Consider curves of embeddings:  $u = w(\infty)$, w_0 $\quad [t_0, \infty) \ni t \to w(t)$

defined by $w^{\#}(t)$ eucl. $= w_0^{\#}$ eucl $+ h(t)$, $w_0 = w(0)$ $\quad (*)$

where $h(0) = 0$, $h(\infty) = h$.

$$\underbrace{\sum_{ij} dx_i \otimes dx_j \left(\frac{\partial w}{\partial x_i} \cdot \frac{\partial \dot{w}}{\partial x_j} + \frac{\partial \dot{w}}{\partial x_i} \cdot \frac{\partial w}{\partial x_j} \right)}_{= 2 dw \odot d\dot{w}} = \sum_{ij} \dot{h}_{ij} \, dx_i \otimes dx_j \quad (**)$$

Assume that $\dot{w}$ is orthogonal to $w(\Sigma)$: $\frac{dw}{dx_j} \cdot \dot{w} = 0 \quad (***)$

We are going to solve $(**)$ & $(***)$.

$$(***) \Rightarrow 0 = \frac{\partial}{\partial x_i}\left(\frac{\partial w}{\partial x_j} \cdot \dot{w} \right) = \frac{\partial \dot{w}}{\partial x_i} \cdot \frac{\partial w}{\partial x_j} + \dot{w} \cdot \frac{\partial^2 w}{\partial x_i \partial x_j}$$

$$\overset{(**)}{\Rightarrow} \quad (\square) \begin{cases} -2 \frac{\partial^2 w}{\partial x_i \partial x_j} \cdot \dot{w} = \dot{h}_{ij} \\ \frac{\partial w}{\partial x_i} \cdot \dot{w} = 0 \end{cases}$$

3.) It would be nice to solve ($\square$) for $\dot{w} = \mathcal{L}\dot{h}_{ij}$ $\Rightarrow$ ODE.

<u>Def:</u> (Gromov) w is called <u>free</u> if in every system of local coords., the $n + \frac{n(n+1)}{2}$ vectors $\frac{\partial w}{\partial x_j}(p)$, $\frac{\partial^2 w}{\partial x_i \partial x_j}(p)$ $(p \in \Sigma, i \le j \in [1 \dots n])$ are lin. indept.

<u>Note:</u> w free $\Rightarrow$ w immersion.
w free + injective $\Rightarrow$ w embedding

<u>Main Thm:</u> Let $w_0 : \Sigma \to \mathbb{R}^N$ be a free C^∞ embedding and let $k \ge 3$.
Then $\exists\, \varepsilon_0 = \varepsilon_0(w_0)$, $\forall h \in C^k$ with $\|h\|_{C^3} < \varepsilon_0$,
$\exists\, \bar{u} : \Sigma \to \mathbb{R}^N$, C^k embedding s.t. $\bar{u}^\#\text{eucl.} = w_0^\#\text{eucl} + h$

Since $\varepsilon_0 = \varepsilon_0(w_0)$, we have to find a good w_0 to deduce embedding thm.

<u>TODO:</u> (1)° Main Thm. $\Rightarrow$ Nash's Thm. (geometric)
(2)° Proof of Main Thm. (analysis)

(1) $\rightsquigarrow$ (i) Fix a free C^∞ embedding w_0, strictly short w.r.t. g
Fix ε_0 s.t. Main thm. applies. $\left(~~[illegible]~~ (\partial_i w_0)(\partial_j w_0)\xi_i\xi_j < g_{ij}\xi_i\xi_j \ \ \forall \xi \neq 0 \right)$
(ii) Build a smooth $\bar{w}$: $h := g - w_0^\#\text{eucl.} - \bar{w}^\#\text{eucl.}$ satisfies $\|h\|_{C^3} < \varepsilon_0$.
(iii) Apply Main thm. (w_0, h) $\Rightarrow$ $\bar{u} : \Sigma \to \mathbb{R}^N$; set $u = \bar{u} \times \bar{w}$ does the job.

[More details - DeLellis]

Note that $(\bar{u} \times \bar{w})^\# = \bar{u}^\# + \bar{w}^\#$.

(4.) If w is free embedding for all t, then in ($\square$) coeff. matrix A has full rank : $A\dot{w} = \binom{\dot{h}}{0}$.

Thus AA^T invertible.

A soln. of ($\square$) is given by $\dot{w} = \underbrace{A^T(AA^T)^{-1}}_{=\mathcal{L}(Dw, D^2w)}\binom{\dot{h}}{0}$.

It follows that we formally have ODE:

$$\dot{w} = \mathcal{L}(Dw, D^2_w)\,\dot{h}$$
$$w(t_0) = w_0 .$$

Problem: Loss of derivatives, $\dot{w}$ depends on Dw, D^2w.

If $w, \dot{h} \in C^k \Rightarrow \mathcal{L}(\dots)h \in C^{k-2}$.

Standard ODE theory (Cauchy-Lipschitz, Picard-Lindelö[f]) won't work.

Nash considers a regularized problem. Introduce a smoothing operator. Key idea - Modify ODE which can be solved.

~~Thm.~~ Lemma: $\exists$ a family $(S_\varepsilon)_{\varepsilon\in(0,1)}$ of linear operators on $C^k(\mathbb{R}^m)$ s.t.

(a) $S_\varepsilon f \in C^\infty(\mathbb{R}^m)$, $\forall f \in C^k$.

(b) $\forall r \geq s, \exists C : \| D^r S_\varepsilon f\|_{C^0} \leq C\varepsilon^{s-r}\|f\|_s$

(c) $\forall r \geq s, \exists C : \| D^r \frac{dS_\varepsilon}{d\varepsilon} f\|_{C^0} \leq C\varepsilon^{s-r-1}\|f\|_s$.

(d) $\forall s > r, \exists C : \| D^r(f - S_\varepsilon f)\|_{C^0} \leq C\varepsilon^{s-r}\|f\|_s$.

Proof: $S_\varepsilon f(x) = \frac{1}{\varepsilon^m}\int_{\mathbb{R}^m} f(x-y)\,\varphi(y/\varepsilon)\,dy$. ∎

Fix $\varphi \in S(\mathbb{R}^m)$. $\int\varphi = 1$, $\varphi \geq 0$.

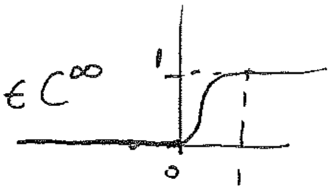
This Lemma extends to fcs on Σ and to tensor fields.

i.) Regularized Problem:

$\left(\mathcal{L}(Dw, D^2w) = \mathcal{L}(w)\right)$

$$\begin{cases} \dot{w} = \mathcal{L}(S_{t^{-1}}\, w(t))\, \dot{h}_w(t) \\ w(t_0) = w_0 . \end{cases}$$

Will require careful choice of $\dot{h}$. Let $\varphi \in C^\infty$



$$h_w(t) = S_{t^{-1}}\left[\varphi(t-t_0)h + \int_{t_0}^{t} 2d\,\left(S_{t^{-1}} w(\tau) - w(\tau)\right) \odot \operatorname{div}(\tau)\,\varphi(t-\tau)\,d\tau\right]$$