Nash R.G. Friday ~~10.30-12.30~~ ~~11-1~~ 11.30-13.00

Two directions for Nash embedding

- C^{α}: constructing strange solutions to nonlinear PDEs

~~C^k, k large:~~ e.g. compactly supp in time to fluid eq^ns on $\mathbb{T}^3$; solutions ~~which~~ don't with prescribed behaviour of kinetic energy.

~~Ex of eq~~ kind of eq^n we care about:
- Buck., Vicol 2017 Nonuniqueness of weak sol^ns for NS
- DeLellis et al. Euler eq^n, Onsager conjecture
- Rüland - Zwichnagel (2016-), nonlinear elasticity

- C^k, k large: Implicit Function Thm in Fréchet spaces, Taylor-made isometric embeddings of manifolds, Nash-Moser-Hörmander iteration.

  Applications:
  - Terry Tao: Blow-up solutions in systems of wave eq^ns. (2016-)

Original Theorems by Nash

Set up: Σ is a manifold if it is a Hausdorff space locally homeomorphic to $\mathbb{R}^N$ and has a ~~cont~~ countable basis of topo.

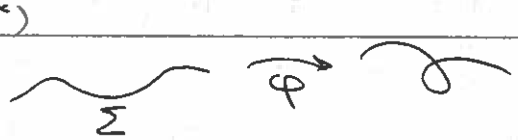
Σ is smooth if $X_p \circ X_q^{-1} \in C^\infty$ $\forall p, q \in \Sigma$.

At every point we have a tangent space, $T_p\Sigma$, and on a Riemannian manifold we have a smooth inner product $g_p : (T_p\Sigma)^2 \to \mathbb{R}$. So in local coord $g_p = \sum_{i,j} g_{ij}\, dx_i \otimes dx_j$.

Nash embedding thms concern PDEs for g_{ij}.

Q: Given abstract manifold Σ, can we embed/immerse it into some $\mathbb{R}^k$?

Immersion: $\varphi: \Sigma \to \mathbb{R}^k$ is immers. if $d\varphi_p$ is injective $\forall p \in \Sigma$ (as a map from $T_p\Sigma$ into $\mathbb{R}^k$)



allows inters.

$\varphi: \Sigma \to \mathbb{R}^k$ is embedding if ~~...~~ φ is bijective immersion.
(no intersections)

Whitney: Σ N-dim manifold $\Rightarrow$
- Σ embeds into $\mathbb{R}^{N}$
- Σ immerses into $\mathbb{R}^{2N-1}$

Nash: isometric embeddings

1) Smooth case: let $k \geq 3$, $N \geq 1$, $M = \frac{N(3N+11)}{2}$.

If (Σ, g) is a closed (= compact, no boundary) Riemmanian manifold of dimension N,
then $\exists$ a C^k isometric embedding into $\mathbb{R}^M$.

Such smooth isometric embeddings are rigid, in the sense that they have many a-priori properties/constraints.

Example: (S^2, g) with C^2-metric and >0 Gauss curvature (differential forms for smoothness of metrics), if there is a $u \in C^{1,\theta}(S^2, \mathbb{R}^3)$ isometric immersion with $\theta > 2/3$
$\Rightarrow u(S^2) =$ boundary of an open convex set.

2) Non-smooth isometric embeddings: "anything works"

Isometric embedding:

$$(\Sigma, g) \xrightarrow{u} (\mathbb{R}^M, \text{eucl})$$

$\Sigma \subset \mathbb{R}^N$

$g = \sum_{ij} g_{ij}\, dy_i \otimes dy_j$ $\qquad$ $\text{eucl} = \sum_{ij} \delta_{ij}\, dx_i \otimes dx_j$

u pulls back eucl to ~~given~~ a metric on Σ

$$u^{\#}\text{eucl} = \sum_{ij} (\partial_i u)\cdot(\partial_j u)\, dy_i \otimes dy_j$$

u is an isometric embedding if $u^{\#}\text{eucl} = g$.

Comparing coef. we get a PDE:

$$\boxed{(\partial_i u)\cdot(\partial_j u) = g_{ij} \quad \forall i,j \le N} \rightarrow \text{nonlinear PDE that Nash studied.}$$

system of $\frac{N(N+1)}{2}$ coupled PDE.

Def. (Short map). Given (Σ, g) R-manifold, $u : (\Sigma, g) \to \mathbb{R}^M$ is short if $u^{\#}\text{eucl} \le g$, in the sense ~~that~~ of inner products on tangent space, i.e.

$$\forall w \in T_p\Sigma \qquad \sum_{ij} (u^{\#}\text{eucl})_{ij}\, w_i w_j \le \sum_{ij} g_{ij}\, w_i w_j \qquad \forall p \in \Sigma$$

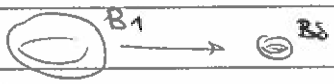
u is strict short if "$<$" holds here $\forall p \in \Sigma\ \forall w \in T_p\Sigma$.

Thm (Nash–Kuiper): (Σ, g) smooth closed N-dim R. manifold, $u(\Sigma, g) \to \mathbb{R}^M$ C^∞ short immersion with $M \ge N+2$.
Then $\forall \varepsilon > 0$ $\exists$ C^1 isometric immersion

$$\tilde{u} : \Sigma \to \mathbb{R}^M \text{ s.t.}$$

$\|u - \tilde{u}\|_{C^0} < \varepsilon$.

Furthermore, if u is an embedding, then $\tilde{u}$ may be chosen as an embedding.

For example: 

is a short map, C^∞

There exist an isometric imersion that does "almost" this.

Corollory (Nash-kuiper + Whitney): Any smooth closed N-dim R-manifold has a C^1 isometric immersion into $\mathbb{R}^{2N-1}$ and a isometric embedding into $\mathbb{R}^{2N}$.

Upshot: • C^3 isometric embedding exist, rigid

• C^1 " " exist and are highly unconstrained.

• C^2 is open.

This shows dependence of the # and propriertiees of solutions of PDE $(\partial_i u)\cdot(\partial_j u) = g_{ij}$ depending on smoothness.

• Valdinoci et al 2016: Every function is some sobolev space is an s-harmonic function, up to ε reminder.

• Euler eq^{n}s: $\forall$ smooth $E = E(t) \geq 0$ $\exists$ a weak sol. ~~u∈L^∞([0,1])~~ $u \in L^\infty([0,1], C^{1/5-\varepsilon}(\mathbb{T}^3))$ of the Euler eq^{n}s with kinetic energy $E(t) = \frac{1}{2}\int |u|^2 dx$.

For smooth sol. $\frac{1}{2}\int |u|^2 dx = \text{const}$.

Onsager conjecture (thm of Isett) (Incompress. Euler eq^n)

For $u \in C^1([0,1]; C^\alpha(\mathbb{T}^3))$ ~~lein~~ with $\alpha < \frac{1}{3}$ kinetic energy cannot be preserved; for $\alpha > \frac{1}{3}$ kinetic energy is preserved.