HERIOT WATT UNIVERSITY

Department of Mathematics Reading Course 2017

Nash Embedding Theorems and Non-uniqueness of Weak Solutions to Nonlinear PDE

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Aim

The aim of this reading group is to understand one of the most celebrated group of results in 20th century mathematics, namely the **Nash Embedding Theorems**. Focus will be on the underlying techniques in the proofs thereof, and their application to the study of nonlinear partial differential equations in mechanics.

Over the course of the semester, aside from delving into the main mathematical ideas behind Nash's insights, we hope to gain some appreciation for the state of **Hadamard's notion of well-posedness** for initial value problems in the modern field of "Analysis of PDE". More specifically, we shall explore the existence, (non-)uniqueness and stability of suitably-defined **weak solutions** for large families of PDE which come from classical mechanics.

Particular emphasis will be placed towards the end of the reading course on the method of **convex integration**, originally introduced by Gromov as a means to establish so-called *h*-principles for PDE.

Target Audience

We imagine that any student or researcher with a serious interest in constructing weak solutions to nonlinear PDE, and the physical applications of differential equations, will find this reading group useful. Those with a strong background in analysis and measure theory ought to benefit

from regular attendance. It might also be helpful to have some basic knowledge of the elements of differential geometry.

Rough Schedule

The following is a rather loosely-constructed list of topics we would like to cover over the course of the semester:

- Introduction and Elements of Differential Geometry **(1 lecture)**:

  Topics: motivation for course; the vague notion of a 'flexible' partial differential equation; the early work of Nash; definition and basic properties of Riemannian manifolds.

- The Nash-Moser Inverse Function Theorem **(2 lectures)**;

  Topics: on the utility of inverse function theorems; the link between the inverse function theorem and isometric embeddings of Riemannian manifolds; calculus on Fréchet spaces; tame Fréchet spaces and tame maps; proof of the Nash-Moser inverse function theorem.

- The Nash-Kuiper C^1 Embedding Theorem **(2 lectures)**;

  Topics: proof of the result; quick discussion of the Nash C^k Embedding Theorem for $k > 1$; counterintuitive consequences of the theorem, including a discussion of the recent numerical work of Borrelli, Jabrane, Lazarus and Thibert.

- The Work of Müller and Šverák **(1 lecture)**;

  Topics: an exposition of the paper of Stefan Müller and Vladimir Šverák, "Convex integration for Lipschitz mappings and counterexamples to regularity".

- Weak Solutions of the Euler Equations and Convex Integration **(3 lectures)**;

  Topics: the work of Scheffer and Shnirelman; the work of Camillo de Lellis and László Székelyhidi on the incompressible Euler equations; Onsager's conjecture.

- Outlook **(1 lecture)**;

  Topics: Current directions in continuum mechanics... TBD!

Of course, depending on the interests of the group members, these topics are subject to change. We welcome all suggestions and modifications to the above rough schedule.

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Date, Time and Location

We shall hold the first reading group on **Tuesday 10th October** from **2PM** to **4PM** in the room **CMT.01** of the Colin Maclaurin Building, Heriot-Watt University. Meetings will continue for 9 weeks thereafter at the same time and place.

Further Information

If you have any questions about the reading group, please send an e-mail to Mark Wilkinson at `mark.wilkinson@hw.ac.uk`.

Image Acknowledgement: The above image depicts the interior of the image of a C^1 isometric embedding of the flat torus in 3-space as calculated in the paper, *Flat tori in three-dimensional space and convex integration* by Borrelli et al., PNAS (2012).