13/03/20

- Abstract error analysis for spectral methods (or other non-Galerkin discretisations of weak formulations).
- Application to Allen-Cahn eqn.
- Comments about special orthogonal polynomials
- Nonlinear continuum dynamics and contact / interface problems.

PDE: $\quad a(u_h, v_h) = F(v_h) \quad \forall v_h \in V_h \quad (*)$

FE: $\quad V_h \subset V$

Find $u_h \in U_h$ st. $(*)$. $\quad U_h \subset U$

V = Dirac - δ's $\longleftrightarrow$ Collocation

$V = U$ $\longleftrightarrow$ Galerkin / standard FE

Thm (i) U, V Hilbert spaces

(ii) $F: V \longrightarrow \mathbb{R}$ linear and cts.

(iii) $a: X \times Y \longrightarrow \mathbb{R}$ sesquilinear form which is

(I) cts: $\quad |a(u,v)| \leq C \|u\|_U \|v\|_V$

injective (II) inf-sup condition: $\quad \inf_{u \in U} \sup_{v \in V} \dfrac{a(u,v)}{\|u\|_U \|v\|_V} \geq \beta > 0$

surjective (III) transposed infsup: $\quad \sup_{v \in V} a(u,v) > 0 \quad \forall u \in U \setminus \{0\}$

Then $\exists!\ u \in U$ s.t. $a(u,v) = F(v) \quad \forall v \in V$.

Corollary. Same for subspaces U_h, V_h: $\quad a(u_h, v_h) = F(v_h) \quad \forall v_h \in V_h$

(Cea's Lemma). In this case $\|u - u_h\|_U \leq \tilde{C} \|u - \tilde{u}_h\|_U \quad \forall \tilde{u}_h \in U_h$,

$\tilde{C} = \tilde{C}(C, \beta, \beta_h)$

Remarks: 1. If $U = V$ this generalises the Lax-Milgram theorem and the classical Cea's Lemma for coercive ~~bili~~ bilinear forms a. Coercivity implies both inf-sup and transposed inf-sup conditions.

2. Cases where standard LM can't be applied for $U = V$.

(i) Helmholtz: $-\Delta u - k^2 u = f$.

(ii) Compact perturbations of coercive problems (ie. addition of lower order terms).
In this case, well posedness is equivalent to ~~the two~~ inf-sup + trans inf-sup.

3. For spectral methods often $U \neq V$, $U_h \neq V_h$.
In the case of coercive a, we do not need to impose an inf-sup or coercivity condition on U_h, V_h:

$$a(u_h, u_h) \geq \alpha \|u_h\|^2 \qquad \forall u_h \in U_h$$

follows from ineq. for the whole space.

For the inf-sup condition we need to check separately for subspaces since $\sup_{V_h} \leq \sup_{V}$ when $V_h \subset V$.

$$\text{inf-sup for } U, V \not\Rightarrow \text{inf-sup for } U_h, V_h.$$

(major issue for Navier-Stokes and Stokes problems).

Stokes: $$\begin{pmatrix} -\Delta & \nabla \\ \nabla\cdot & 0 \end{pmatrix} \begin{pmatrix} u \\ p \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix}.$$

zero

Proof of Thm. See notes on webpage.

Proof of Cea's Lemma.

Denote by P_{V_h} the orthogonal projector $P_{V_h}: V \to V_h$,

and by $R(u)$ the operator $R: U \to V$

$$a(u,v) = (R(u), v)_V \qquad \forall v \in V.$$

NOTE: R well defined. ~~[illegible]~~.

eg. $a(u,v) = \int \nabla u \cdot \nabla v$, $R(u) = u$, $U, V = H_0^1$.

$$Q := P_{V_h} R : U \xrightarrow{R} V \xrightarrow{P_{V_h}} V_h$$

Define $S: U_h \to V_h$ by $(Su_h, v_h)_V = a(u_h, v_h)$ $\quad \forall u_h \in U_h, v_h \in V_h$

Claim. $Su_h = Qu_h \quad \forall u_h \in U_h$.

Pf. $(P_{V_h} R u_h, v_h)_V = (Ru_h, P_{V_h} v_h)$

$= (Ru_h, v_h)$

$= a(u_h, v_h)$ $\square$

$$\Rightarrow \quad S = Q|_{U_h}.$$

For the RHS we ~~[illegible]~~ let $v_{0,h} \in V_h$ be st.

$$F(v_h) = (v_{0,h}, v_h)_{V_h}, \quad \forall v_h \in V_h.$$

Then since $a(u, v_h) = F(v_h)$, $\forall v_h \in V_h$ we have

$$v_{0,h} = P_{V_h} R u.$$

Further, $$u_h = S^{-1} P_{V_h} R u ,$$

where S is invertible by existence of sol^{n} to eq^{n} on U_h, V_h.

Then

$$u_h - u = S^{-1} P_{V_h} R u - u . \qquad (*).$$

NOTE. $$S^{-1} P_{V_h} R \Big|_{U_h} = I \Big|_{U_h} .$$

Then $$(*) = \left(S^{-1} P_{V_h} R - I \right)(u - \tilde{u}_h) \qquad \forall \tilde{u}_h \in U_h .$$

Moreover, the operator $S^{-1} P_{V_h} R - I$ is bounded so

$$\boxed{\| u - u_h \|_U \leqslant \tilde{C} \, \| u - \tilde{u}_h \| \qquad \forall \tilde{u}_h \in U_h}$$

where $\tilde{C} = \| S^{-1} P_{V_h} R - I \|$. ■

Further extensions: PDE. Find u s.t. $a(u,v) = F(v)$.

FE. Find $u_h \in U_h$ s.t. $a_h(u_h, v_h) = F_h(v_h) \quad \forall v_h \in V_h$.

where a_h and F_h are discretisations of a_h and F_h.

eg. if you evaluate $\int_\Omega$ by a quadrature rule.

(See Strang's Lemma)

How to choose spectral methods for a real-world problem

Allen-Cahn eq. from materials science.

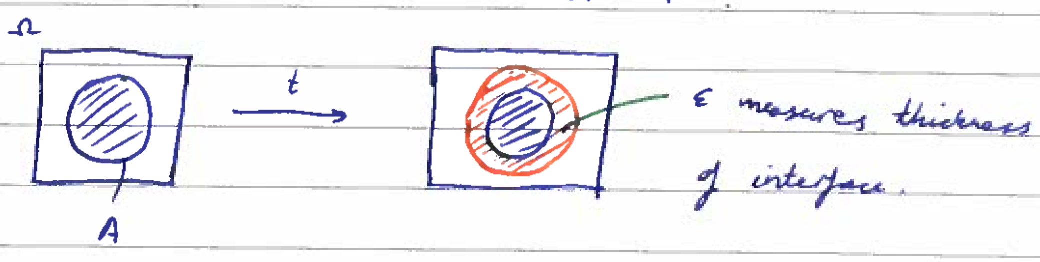
$$\underbrace{\partial_t u}_{} - \varepsilon^2 \Delta u + u^3 - u = 0$$

Replace by finite difference approx.

$u(t=0)$ given. $\qquad \Omega = [-1,1]^2$

periodic B.C.

NOTES:
- Think of icecream melting in your mouth.
- ε ~~xxx~~ measures diffusion. ~~xxx~~
- eg. initial condition $u \equiv 0$ on $\Omega \setminus A$
  $u \equiv 1$ on A



Spectral method in space.

$$u_N(x,y,t) = \sum_{k,l=-\frac{N}{2}}^{\frac{N}{2}} \tilde{u}_{k,l}(t)\, e^{i(kx+ly)\pi}$$

$\rightsquigarrow$ $(N+1)$-dimensional ODE for $\tilde{u}(t)$.

Calling u_N^m the approx. of u_N at $t = t_m = m\,dt$.

We would then solve the system

implicit method

$$\underbrace{\frac{3u_N^{m+1} - 4u_N^m + u_N^{m-1}}{2dt}}_{\text{discrete time derivative}} - \varepsilon^2 \Delta u_N^{m+1} + (u^3 - u)^{m+1} = 0$$

~~xxx~~ Choose u_N^1 or use only 1st order discretisation.

If this is m then you would get a (explicit) recursion relation whereby u^{m+1} depends on $\Delta^{(m+1)} u_N^0$. Would need very strong info on initial data. Problems at high freq.

With $m+1$ you make things smoother because you apply Δ^{-1} to initial data.

<u>Algorithm.</u>

U_N^0 given

$\downarrow$

write down time independent problem involving U_N^0 and U_N^1.

$\downarrow$

Solve time indep. problem using Galerkin or Spectral method

$\downarrow$

OUTPUT: U_N^1

$\downarrow$

OUTPUT: $(U_N^0, U_N^1, U_N^2, \ldots, U_N^{M-1}, U_N^M)$.

<u>Q^{n}.</u> What do you do with the non-linear part?

$\sin^3(x)$ has highest frequency 3.

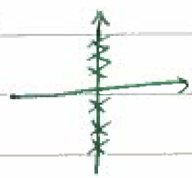
Need to decide what to do considering:

- facility of solution
- accuracy of solution / reliability of sol^{n}.
- fast convergence at each time step / for whole scheme.

<u>NOTE:</u> Wave equation: $\partial_t^2 u = (\text{coercive})\, u$

eivalues

$$\partial_t \begin{pmatrix} u \\ v \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 \\ \text{coercive} & 0 \end{pmatrix}} \begin{pmatrix} u \\ v \end{pmatrix}$$



NOT COERCIVE.