LECTURE 8

25/03/2020

SPECTRAL METHODS

The ingredient of the above methods

Different weak formulations of PDE

- **Galerkin:**

Find $u \in H$ $\int \nabla u \nabla v = \int f v \quad \forall v \in H$

Note u,v belong to the same space H.

- **Weighted residual:**

Find $u \in U$ $\int (\Delta u - f) w v = 0 \quad \forall v \in V$

Note u,v belong to different spaces.

Spectral collocation

$V = \text{span}\{\delta x_i\}$ $i \in \{1, .., N\}$

$\forall i \; f(x_i) = \Delta u(x_i)$

$U = \text{span}\{\sin(kx) \;\; k \in \{1, ..., N\}\}$

$x_j = jh \quad h = \frac{2\pi}{N}$

How accurately you can calculate a smooth function $u \in C^{\infty}$

Last time:

If you use this approximation

eg. $u(x) = \ln(2+\sin(x)) \simeq \sum_{i=0}^{N-1} u(x_i)\Lambda_i(x)$

$$\Lambda_i(x_j) = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases} = \frac{1}{N}\sin\left(N\frac{x-x_j}{2}\right)\cot\left(\frac{x-x_j}{2}\right)$$

Calculate

$$\partial_x u \simeq \sum_{i=0}^{N-1} u(x_i)\partial_x \Lambda_i =$$

$$= \frac{1}{N}\sin\left(N\frac{x-x_j}{2}\right)\cot\left(\frac{x-x_j}{2}\right) \in U$$

$\simeq D\tilde{u}$ where $\tilde{u} = \begin{pmatrix} u(x_0) \\ \vdots \\ u(x_{N-1}) \end{pmatrix}$

The accuracy was really good but recall the matrix D is dense

This gives very accurate results, even for small N. Why?

Can we use this to solve PDE?

② We can use these formulations, if we use appropriate U, V

Today's Lecture

Main Theorem about Fourier interpolation.

- Define the interpolation operator:

$I_N u(x) := \sum_{i=0}^{N-1} u(x_i) \Lambda_i(x)$

Then, for $u \in H^m(0,2\pi) = \{u : (0,2\pi) \to \mathbb{R} : u, \partial_x u, \partial_x^m u \in L^2\}$

$$\boxed{\|\partial_x^\ell (I_N u - u)\|_{L^2(0,2\pi)} \leq C_{m,\ell} \|\partial_x^m u\|_{L^2}}$$

If $u \in C^\infty(0,2\pi) \Rightarrow$ you can choose m arbitrary
(m smooth as you want).

$$\boxed{\|\partial_x (I_N u - u)\|_{L^2} \leq C_m N^{1-m}}$$

The error will decrease fast than N^{-m} $\forall m$.

Using that $H^1(0,2\pi) \subseteq C(0,2\pi)$
and recall Sobolev inequality from previous lect.

$$\boxed{\max_x |\partial_x^\ell (I_N u(x) - u(x))| \leq C N^{\ell+1-m} \|\partial_x^m u\|_{L^2}}$$

$\Rightarrow$ The pointwise error of using the trigonometric interpolation $\to 0$ faster than N^{-m} $\forall m$.

Proof

Use Fourier techniques

standard Fourier coefficient:

$$\hat{u}_k = \frac{1}{2\pi}\int_0^{2\pi} u(x) e^{-ikx}\,dx \quad k\in\mathbb{Z}$$

Coefficients of $I_N u(x) = \sum_{k=-N}^{N} \tilde{u}_k e^{ikx} \quad k\in\mathbb{Z}$

Observation (Exercise)

$$\tilde{u}_k = \frac{1}{2Nc_k}\sum_{j=0}^{2N-1} u(x_j) e^{-ikx_j} \quad k\in\{-N,-N+1,\dots N-1,N\}$$

where $c_k = 1$ if $|k|\leq N$
$c_k = 2$ if $|k|\leq N$

Claim: $c_k \tilde{u}_k = \hat{u}_k + \sum_{|p|>0}^{\infty} \hat{u}_{k+2pN}$

Indeed, $u(x_j) = \sum_{|p|=0}^{\infty} \hat{u}_p e^{ipx_j}$

so that from the observation

$$c_N \tilde{u}_k = \frac{1}{2N}\sum_{j=0}^{2N-1}\left\{\sum_{|p|>0} \hat{u}_p e^{i(p-k)x_j}\right\}$$

$$= \frac{1}{2N}\sum_{|p|>0} \hat{u}_p \left(\sum_{j=0}^{2N-1} e^{i(p-k)x_j}\right) = \begin{cases} 1 \Leftrightarrow p-k = 2lN \\ \quad \exists\, l\in\mathbb{Z} \\ 0 \end{cases}$$

$$\Rightarrow c_k \tilde{u}_k = \sum_{|p|=0}^{\infty} \hat{u}_{k+2pN} = \hat{u}_k + \sum_{|p|>0} \hat{u}_{k+2pN}$$

The piece of algebra shows the claim.

Remaining steps:

(1) Show that $\hat{u}_K$ decreases very quickly:

$$\underbrace{\|u-\sum_{K=-N}^{N}\hat{u}_K e^{iKx}\|}_{\text{Nth Fourier approximation}}{}_{H^j(0,2\pi)} \le C_N^{j-m}\|u\|_{H^m} \quad \forall\, m\ge j$$

(2) Use claim to prove main Theorem for $\tilde{u}_K \Leftrightarrow I_N u$

Proof of (1)

Write down norms in terms of Fourier coeffic.

$$\|v\|^2_{H^j(0,2\pi)} := \|v\|^2_{L^2} + \|\partial^j v\|^2_{L^2}$$

Parseval's identity

$$\sum_K |\hat{v}_K|^2 + \sum_K |(\widehat{\partial^j v})_K|^2 = \sum_K (1+|K|^2)|\hat{u}_K|^2$$

From definition of Fourier coefficient

$$(\widehat{\partial^j v})_K = (\pm iK)^j\, \hat{v}_K$$

To show:

$$\sum_{|k|>N} (1+|k|^{2j})|\hat{u}_k|^2 \leq CN^{j-m} \sum_k (1+|k|^{2m})|\hat{u}_k|^2$$

For the left hand side,

$$\sum_{|k|>N} (1+|k|^{2j})\,|\hat{u}_k|^2 \leq N^{2(j-m)} \sum_{|k|>N} \frac{|k|^{2m}(1+|k|^{2j})|\hat{u}_k|^2}{|k|^{2j}}$$

$$\overset{|k|>N}{\leq} 2N^{-2(j-m)} \sum_{|k|>N} |k|^{2m}\,|\hat{u}_k|^2$$

$$\leq 2N^{2(j-m)} \underbrace{\sum_k (1+|k|^{2m})\,|\hat{u}_k|^2}_{=\,\|u\|^2_{H^m}}$$

$\Rightarrow$ 1

For 2 one uses claim to translate decay of $\hat{u}_k$.

Refinement of main Theorem

If u extends to an analytic function then

$$\|u - I_N u\|_{L^2} \leq e^{-\tilde{C}N} C \quad \text{for } C>0$$

$$\max_x |u(x) - I_N u(x)|$$

error decays exponentially fast

Hope: If the special method for PDE satisfies some version of Cea's Lemma ie.

$\|FE\ error\| \leq C_1 \|error\ of\ best\ approx\|$

Then the fast best approximation implies fast convergence of the numerical method.

Question: Does Cea's lemma hold?

Need version for $a: U \times V \to \mathbb{R}$ bilinear with U, V different (Hilbert) spaces

Theorem (Babuska) Generalization of Lax-Milgram

U, V Hilbert spaces

$a: U \times V \to \mathbb{R}$ bilinear form: $|a(u,v)| \leq C_1 \|u\|_U \|v\|_V$

$F: V \to \mathbb{R}$ linear continuous.

inf-sup condition: (generalizes coercivity)

$$\exists \alpha > 0: \inf_{u \in U} \sup_{v \in V} \frac{|a(u,v)|}{\|u\|_U \|v\|_V} \geq \alpha$$

also called Ladyzhenslagen Rabuska Brezzi condition

transposed inf-sup condition:

$$\sup_{u \in U} |a(u,v)| > 0 \quad \forall v \in V$$

Then:

there exists a unique solution $u \in U$ to $a(u,v) = F(v)\ \forall v \in V$ and $\|u\|_U \leq \alpha^{-1} \|F\|_{V'}$

If $U_n \subseteq U$ $V_n \subseteq V$ st inf-sup condition and transposed inf-sup condition still hold for U_n, V_n then also the

Problem:

Find $u_n \in U_n$ s.t. $a(u_n, v_n) = F(v_n)$ $\forall v_n \in V_n$ admits a unique solution $u_n \in U_n$

Cea's Lemma

If U, V U_n, V_n satisfy the above condition then $\boxed{\|u - u_n\|_U \leq \tilde{C} \|u - \tilde{u}_n\| \; \forall \tilde{u}_n \in U_n}$

Numerical error $\leq C \cdot$ Best approximation error