LECTURE 7 18/02/2020

Last week (Lecture 6)

Finite Difference Methods

$u:(0,1) \to \mathbb{R}$ $h>0$ small

$$\partial_x u(x) \simeq \frac{u(x+h)-u(x)}{h}$$

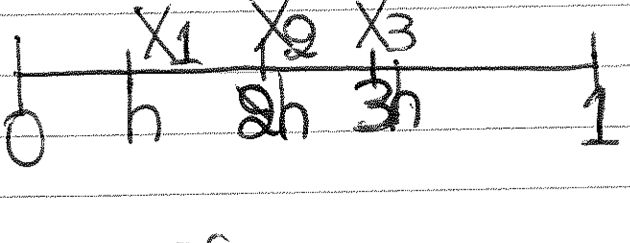
$$\partial_x^2 u(x) \simeq \frac{u(x+h)-2u(x)+u(x-h)}{h^2}$$

$-\partial_x^2 u = f$ in $(0,1)$ $\Longrightarrow$
$u(0)=u(1)=0$

$A\vec{c} = \vec{F}$ where

$$A = \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & -1 & \\ & & & \ddots & -1 \\ & & & -1 & 2 \end{bmatrix}$$

$c_j \simeq u(jh)$ $f_j = f(jh)$



Note: If you are sure that your solution is smooth finite difference methods are really fine otherwise it may not be

Star Lemma

Suppose $(\alpha_i, p_i)_{i=0}^{K}$ are such that

- $\alpha_i < 0$ for $i = 1, \dots, K$
- $\alpha_0 + \alpha_1 + \dots + \alpha_K \geq 0$
- $\alpha_0 p_0 + \alpha_1 p_1 + \dots + \alpha_K p_K \leq 0$
- either $p_0 \geq 0$ or $\alpha_0 + \dots + \alpha_K = 0$

Then if $p_0 = \max p_i \Rightarrow p_0 = p_1 = \dots = p_K$

If we apply Star Lemma to $A\vec{c} = F$ with

$A\vec{c} = F$ with $F_i \leq 0$ for all i

and if c_j is the maximum of the c's then either
- j corresponds to a boundary node $j = 0$ or N
- c_j is constant: $c_0 = c_1 = \dots = c_N$

Theorem (Discrete Maximum Principle)

a) Let $A\vec{c} = \vec{F} \leq 0$ be the finite
be the finite difference discretization

of $-\partial_x^2 u = f$ in $(0,1)$
$u(0), u(1)$ given

Then, $\max\limits_i c_i = \max\{u(0), u(1)\}$

b) Assume the max is attained at an interior node
Then, $c_0 = c_1 = \dots = c_N$

Look slides on the webpage for the proof of Theorem
Convergence of Finite difference approximations

Error equation

$-\partial_x^2 u(x) = f(x)$

Discretization of Laplacian

$-\partial_h^2 u_h(x_j) := -\frac{u_h(x_{j+1}) - 2u_h(x_j) + u_h(x_{j-1})}{h^2} = f(x_j)$

- Assume $u(x_j) = u_h(x_j)$ for $x_j \in \partial\Omega$

exact discrete

$-\partial_h^2 (u - u_h)(x_j) = -\partial_h^2 (x_j) - (A\vec{c})_j$

$(A\vec{c} = f)$ $\quad = -\partial_h^2 (x_j) - f(x_j)$

$(-\partial_x^2 u = f)$ $\quad = -\partial_h^2 (x_j) + \partial_x^2 u(x_j) =: -r_j^h$

"unknown" side "Known" side

The error $\eta(x_j) = u(x_j) - u_h(x_j)$ satisfies

$\boxed{\partial_h^2 \eta(x_j) = -r_j^h} \ \forall j$

we want to conclude solving this eq.

where r_j^h: remainder

Convergence:

If $\boxed{r_j^h \xrightarrow{h \to 0^+} 0}$ and $\boxed{(\partial_h^2)^{-1}}$ bounded as $h \to 0^+$

consistency stability

Look slides

Today 18/2/2020 Convergence of FE

Spectral Methods

Look for **References** on the webpage

i.e. - Shen Tang Wang, Spectral Methods, Springer 2011

- Trefethen Spectral Methods in Matlab SIAM 2000

Recall: General set up for Finite Elements (FE)

- H: Hilbert space $H_n \subseteq H$ finite dimens subspace
- $a: H \times H \to \mathbb{R}$ bilinear form
  - continuous: $|a(u,v)| \leq C \|u\|_H \|v\|_H$
  - coercive : $a(u,u) \geq \alpha \|u\|_H^2$
- $f: H \to \mathbb{R}$, linear, continuous: $|f(v)| \leq C \|v\|_H$

Weak formulation of PDE:

$u, v \in H$ in same space

Find $\underline{\underline{u \in H}}$ st. $a(u,v) = f(v)$ $\forall$ $\underline{\underline{v \in H}}$

Discretization

suitable choice gives good approxim

FE equation: $a(u_n, v_n) = f(v_n)$ $\forall$ $v_n \in H_n \subseteq H$

Key example: Ω

$$a(u,v) = \int_\Omega [(A(x)\nabla u(x)) \cdot \nabla v(x) + b(x)\nabla(u(x)) v(x) + c(x)u(x)v(x)$$

Starting point:

Strong formulation:

$$-\Delta u = f \text{ in } \Omega$$
$$u = 0 \text{ on } \partial\Omega$$

Weighted weak formulation: Multiply by $v(x)w(x)$ and

$$\int_\Omega (\Delta u(x) + f(x))\, v(x)\, w(x)\, dx = 0 \quad \forall\, v \in V$$

where v: test function
w: weight function > 0

Integrating by parts

Standard variational formulation:

$$-\int_\Omega \nabla u \cdot \nabla(vw) = \int_\Omega f\, vw \quad \forall\, v$$

and again by integration by parts

$$-\int_\Omega u \Delta(vw) = \int_\Omega f\, vw \quad \forall\, v$$

The above is called:
Ultra variational formulation
distributional formulation

All of these have the form

Find $\underline{\underline{u \in U}}$ s.t. $\forall\, \underline{\underline{v \in V}}$ $a(u,v) = f(v)$

NB: different spaces

Numerical discretization

$U \supseteq U_n$ finite dimensional subspace

$V \supseteq V_n$ finite dimensional subspace

Numerical approximations by quadrature rule for $\int_\Omega$

$u \to u_n$
$f \to f_n$

Before discussing the above, we give some examples

Choose U_n, V_n

1) $\sin(kx)$, $\cos(kx)$, e^{ikx} $k \in \mathbb{Z}$

Approximate $u = \sum c_k e^{ikx}$ by its Fourier series

2) Chebyshev Polynomials: $T_k(x)$ $k \in \mathbb{N}$

Look Trefethen $\to$ chebfun

3) Legendre polynomials: $L_k(x)$ $k \in \mathbb{N}$ $\square$

U_n and V_n need not coincide.

Eg. $V_n = \text{span}\{\delta x_j : j = 1, \dots N\}$

This means: (weighted weak formulation)

$$0 = (\Delta u_n(x_j) + f(x_j)) w(x_j)$$

i.e. I'm satisfying the PDE in the points $x_j \in \Omega$

Spectral collocation method

$U_n = \text{span}\{\sin(kx) : k \in \{1, \dots N\}\}$

$V_n = \text{span}\{\delta x_j \; j = 1, \dots N\}$

Use this for $\boxed{\begin{array}{l} -\partial_x^2 u = f \text{ in } (0, 2\pi) \\ u(2\pi) = u(0) \text{ periodic boundary c.} \end{array}}$

i.e. Find solution $U_n(x) = \sum_{k=1}^{N} C_k \sin(kx)$ s.t.

$-\partial_x^2 U_n = f(x_j)$

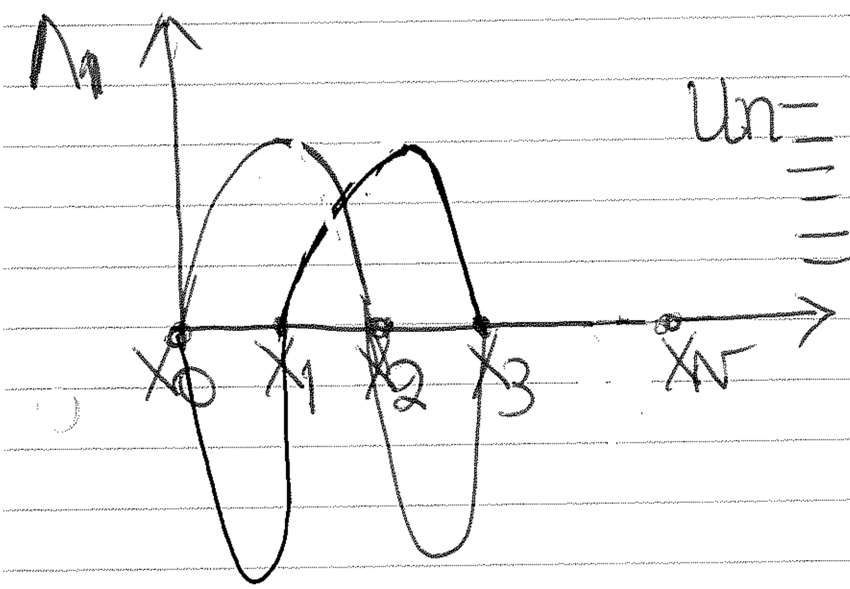
It's better to write as linear combinations

More convenient: Basis of hat functions

$x_j = jh \quad h = \frac{2\pi}{N}$

$$\Lambda_k(x_j) = \begin{cases} 1 & \text{if } k = j \\ 0 & \text{if } k \neq j \end{cases}$$

$$= \frac{1}{N} \frac{\sin(N(x - x_k)/2)}{\sin((x - x_k)/2)} \cos\left(\frac{x - x_k}{2}\right)$$



$U_n = \sum_{k=1}^{N} C_k \sin(kx)$

$= \sum_{k=1}^{N} a_k \Lambda_k(x)$

Discretizing derivatives

$$\partial_x u_h(x_j) = \sum_{k=1}^{N} d_k \, \overbrace{\left[\partial_x \Lambda_k(x_j)\right]}^{\text{you can precompute}}$$

$$\stackrel{*}{=} \begin{cases} 0 & \text{if } j=k \\ \frac{(-1)^{k+j}}{2} \cot\left(\frac{(j-k)\pi}{N}\right) & \text{if } j \neq k \end{cases}$$

The (*) equality can be shown (as an exercise)

$$= D\vec{d} \qquad \text{where } D \text{ is dense matrix}$$

$$\begin{bmatrix} 0 & \cot(h/2) & \cdots & \\ -\cot(h/2) & 0 & \ddots & \\ \cot(h) & \ddots & 0 & \ddots \\ & \ddots & \ddots & 0 \end{bmatrix}$$

You can compare the accuracy of this formula $\partial_x u_h(x_j)$

Choose piecewise polynomials

$$u_h(x) = \sum_{k=1}^{N} c_k \sin(kx) = \sum_{k=1}^{N} a_k \Lambda_k(x)$$

Discretize numerically $\ln(2+\sin(x))$ and interpolate by high order polynonomials or $\cos x$, $\sin x$.

- You can also do the same approach for 2nd derivatives

$$\text{PDE} \quad -D^2 \vec{d} = \vec{f} \qquad \text{where } D^2 \text{ dense matrix}$$

Spectral allocation method for $-\partial_x^2 u = f$

We could equally well use:

$U_n = \text{span}\{\sin(kx) \quad k=1,\dots N\}$

$\parallel$

V_n for the variational formulation:

$$\int (\partial_x u)(\partial_x v) = \int f v \quad \forall v$$

Spectral finite element method

different choice of V_n and of bilinear form

Advantage of this form:

We know all the theory:

$$\|u - U_n\|_{H^1} \leq C \inf_{V_n \in U_n} \|u - V_n\|_{H^1}$$

next week

ie FE error $\leq$ C (error of best approximation)

Disadvantages:

- Have to compute $\int_\Omega$
- It is more expensive to set up linear algebra

More spectral methods

- $U_n \neq V_n \Rightarrow$ Petrov-Galerkin method
  collocation is an example

Note: Always need dim $U_n = V_n$ to hopefully get a unique numerical solution

Numerical integration of bilinear form
U_n, V_n anything $a(u_n, v_n) = \int_{\Omega} (\nabla u_n)(\nabla v_n)$
is approximated by
numerical quadrature rule
(be chosen carefully)

Next week:

- What kind of test function v is allowed and give us a good approximation and fast convergence
- Convergence of spectral allocation and other methods
- Compatibility conditions so solutions can make sense

Announcement for 1rst year PhD students

You should give talks on a topic you would prefer on 17th March 10^{30}
Pizza will also be provided ☺