Lecture 11/02/2020

Framework for Finite Elements

- **Weak Form of the PDE:**

$$a(u,v) = f(v) \qquad \forall v \in H$$

- **FE equation:**

$$a(u_h, v_h) = f(v_h) \qquad \forall v_h \in H_h \subseteq H$$

- **Here:**
  - H Hilbert space
  - $H_h \subseteq H$, finite dimensional subspace
  - $a : H \times H \longrightarrow \mathbb{R}$; bilinear form
    - continuous: $a(u,v) \leq C \|u\|_H \|v\|_H$
    - coercive: $a(u,u) \geq \alpha \|u\|_H^2$
  - $f : H \longrightarrow \mathbb{R}$; linear, continuous
    - $|f(v)| \leq C \|v\|_H$ $- C\|u\|_{L^2}^2$ lower order terms

- **Example:** $a(u,v) = \int_\Omega \Big[(A(x) \nabla u(x)) \cdot \nabla v(x)$

$$+ \underbrace{b(x)}_{\text{lower order terms}} \nabla(u(x))\, v(x)$$

$$+ c(x)\, u(x)\, v(x) \Big]$$

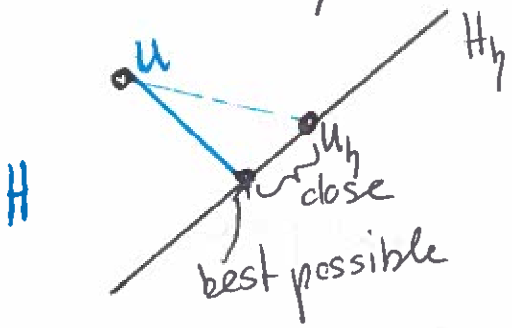
$\iff$ $-\nabla \cdot (A(x) \nabla u(x)) + b(x) \nabla u(x) + c(x)\, u(x)$

Cea's Lemma

This approximation is (almost) optimal.

$$\|u-u_h\|_H \le \frac{C}{\alpha}\|u-v_h\|_H \qquad \forall v_h \in H_h.$$

- Here $\frac{C}{\alpha}$ is independent of H_h, C is from the continuity condition; α is from coercivity
- $\|u-v_h\|_H$ is the error of the best possible approximation in H_h.



For reasonable choices of H_h we will prove convergence of FEM.

Convergence if $H = \overline{\bigcup_{h\to 0} H_h}$.

In particular, if H_h = piecewise polynomials on a mesh of meshsize h.

Proof:

- Basic Observation: Galerkin Optimality

$$\forall v_h \in H_h: \ a(u-u_h, v_h) \underset{\text{linearity}}{=} a(u,v_h) - a(u_h,v_h)$$

$$= f(v_h) - f(v_h) \quad \text{(from the weak form of the PDE)}$$

- **From coercivity of a:**

$$a(u-u_h, u-u_h) \geq \alpha \|u-u_h\|_H^{\cancel{2}}$$

$$a(u-u_h, u-v_h) = a(u-u_h, u-v_h+v_h-u_h) =$$
$$+ \underbrace{a(u-u_h, v_h-u_h)}_{=0 \text{ by Galerkin orthogonality}}$$

$$= C\cancel{\|u-u_h\|_H}\|u-v_h\|_H$$

$$\Rightarrow \quad \|u-u_h\|_H \leq \frac{C}{\alpha}\|u-v_h\|_H \qquad \forall v_h \in H_h \qquad \boxtimes$$

- **Lax-Milgram:** H Hilbert space.

  If $a: H \times H \to \mathbb{R}$, bilinear form, continuous and coercive and
  - $f: H \to \mathbb{R}$, linear and continuous,

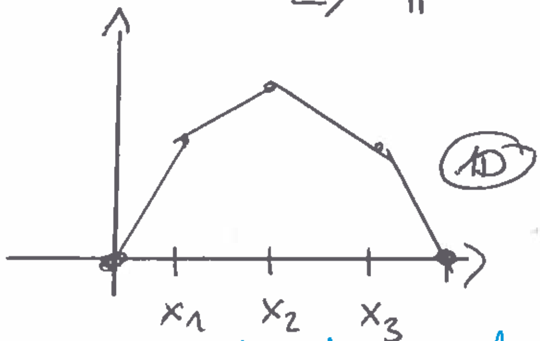
  then

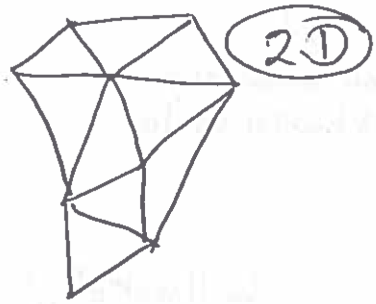
$$\exists!\; u \in H : \; a(u,v) = f(v) \quad \forall v \in H.$$

Lax-Milgram shows $\exists!$ solution to both the PDE and FE equation (with H and H_h, respectively).

Fun with H_h: What are suitable choices of H_h?

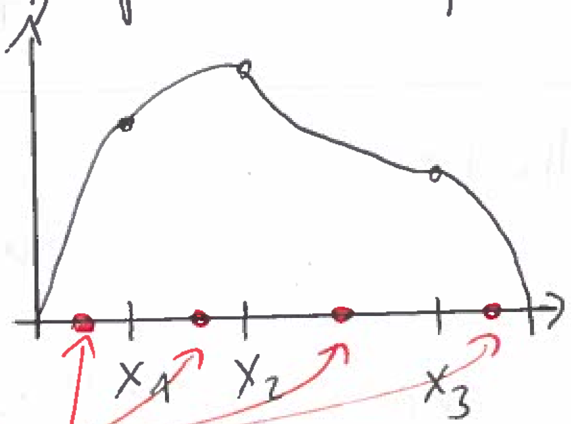
- Classical: H_h = Piecewise linear functions on a mesh
  $\Rightarrow \mathbb{P}^1$



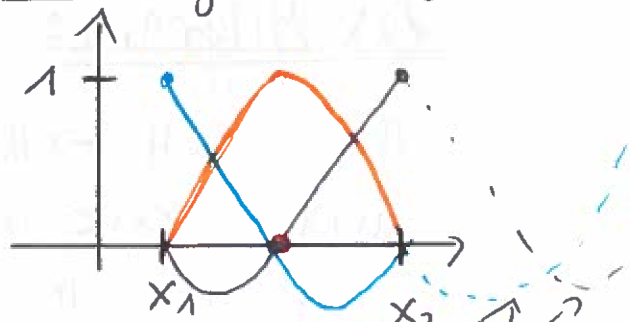


uniqely determined by end points

- H_h = piecewise quadratic functions on a mesh
  $\Rightarrow \mathbb{P}^2$



3 Basis functions for $\mathbb{P}^2$



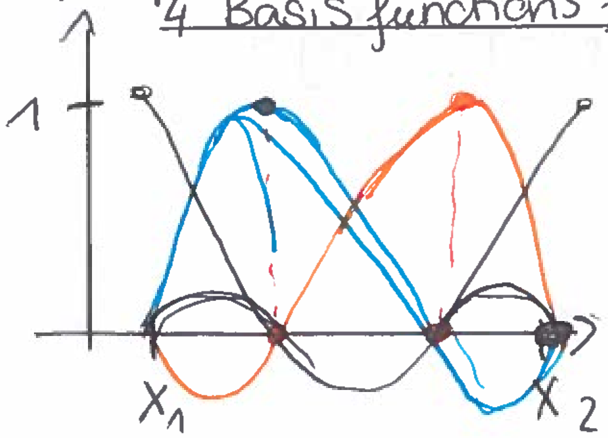
add midpoints.

not uniquely determined by endpoints
↳ add midpoints

supported in two interval for continuity (avoid jumps)

- H_h = piecewise cubic functions on a mesh $= \mathbb{P}^3$

4 Basis functions for $\mathbb{P}^3$



- need 2 auxilliary points
- cubic polynomial is uniquely defined by 4 points

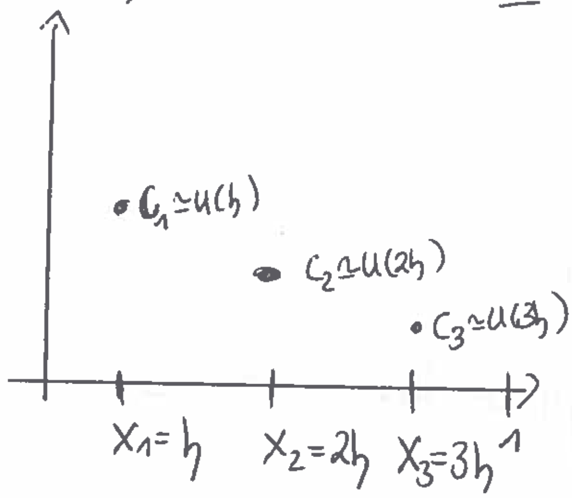
- generally: $\mathbb{P}^r$ can be done using $p+1$ basis (s) functions.

- More examples on the webpage: (see website for detail
  - → discussed how many points are needed in 2D:
    - $\mathbb{P}^1$: 3pts
    - $\mathbb{P}^2$: 6pts
    - $\mathbb{P}^3$: 10pts
    - $\mathbb{P}^p$: $\frac{(p+1)(p+2)}{2}$ pts
  - → discussed how to prove these ideas for higher orders
  - → discussed that triangles are not the only shape we can model
    - ↳ Pegasus mesh...
  - → looked at "Periodic Table" (pw-polynomials in 1D, 2D, 3
    - ↳ can model other shapes, prescribe derivatives at nodes, and other things...
    - ↳ basis functions do not need to be polynomials...

Finite Difference Method

Main Content can be found on the website.
Here are the ideas discussed on the board:

We have: $A\underline{c} = \underline{F}$ ← discretized system, $c_j \simeq u(jh)$



- $-\partial_x^2 u = f$ convex, maximum is on the boundary

⇒ does the discrete version also satisfy this?

- Star Lemma (full Statement in notes)

Proof: Consider

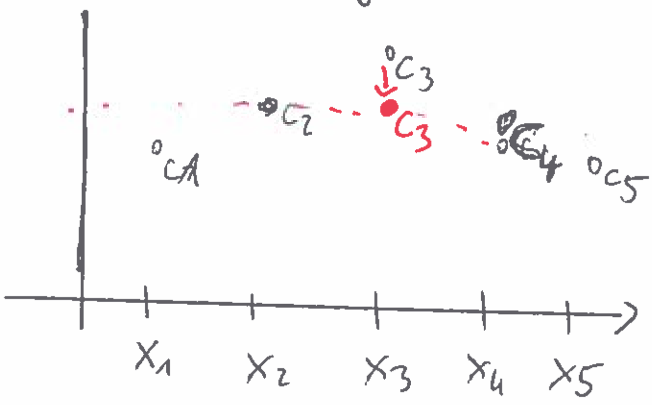
$$\sum_{i=1}^{k} \underbrace{\alpha_i}_{<0} \underbrace{(p_i - p_0)}_{\leq 0 \text{ since } p_0 = \max p_i \text{ by assumption}} \geq 0 \quad (a)$$

$$= \sum_{i=1}^{k} \alpha_i (p_i - p_0) - \alpha_0 \underbrace{\cancel{(p_0 - p_0)}}_{=0}$$

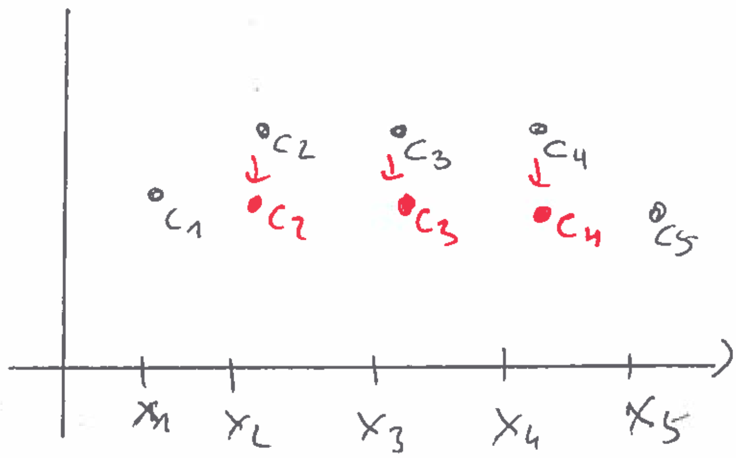
$$= \underbrace{\sum_{i=1}^{k} \alpha_i p_i}_{\leq 0} - \underbrace{p_0 \sum_{i=1}^{k} \alpha_i}_{\geq 0} \leq 0 \quad (b)$$

Since the sum is (a) ≥ 0 and (b) ≤ 0,
we conclude that $\sum_{i=1}^{k} \alpha_i (p_i - p_0) = 0$, and so
$\alpha_i (p_i - p_0) = 0 \;\; \forall i$ and finally since $\alpha_i \neq 0$, we get $p_i = p_0 \; \forall i$.

We have either the maximum is on the boundary or the solution is constant.
Argument is stepwise:



if we find that c_3 is max, by the ~~Star~~ lemma we find that c_2, c_4 are maximum as well.



by the same process we move c_2, c_3, c_4

Finally, On the website, there are some slides on the finite difference error.

Main difference to keep in mind between FE & Finite Differences is that in finite differences we need u to be twice differentiable.
This is not necessary in FE.