Implementation

Problem:

$$-\Delta u = f \quad \text{in } \Omega$$
$$u = u_D \quad \text{on } \partial\Omega_D$$
$$\partial_n u = g \quad \text{on } \partial\Omega_N$$

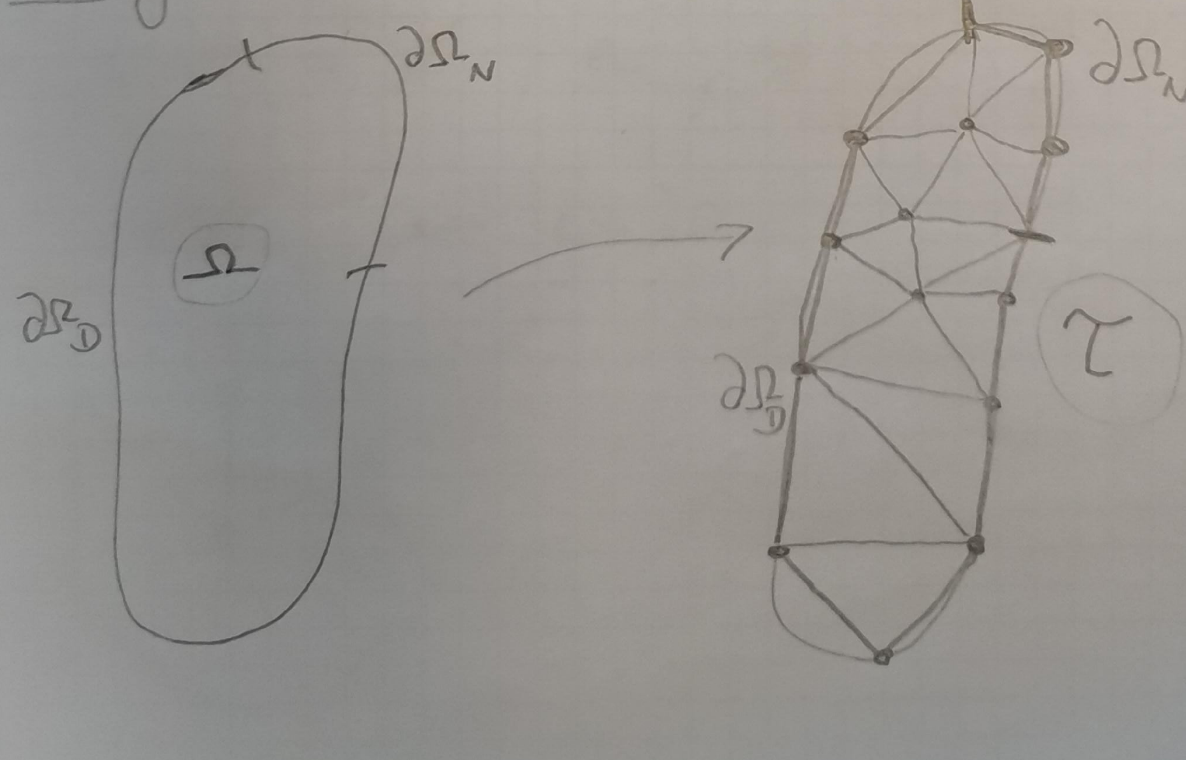
From theoretical standpoint: $u_D \in H^{\frac{1}{2}}(\Gamma_D) \longleftrightarrow u_D \in H^1(\Omega)$.

Weak Form: Find $u_0 \in H^1(\Omega)$ such that

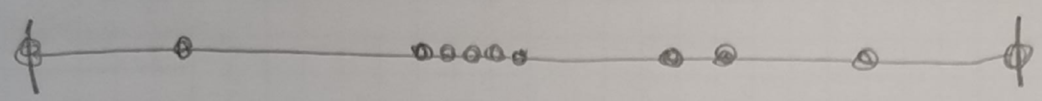
$$\int_\Omega \nabla u_0 \cdot \nabla v \, dx = \int_\Omega f v \, dx + \int_{\partial\Omega_N} g v \, dS - \int_\Omega \nabla u_D \cdot \nabla v \, dx$$

for all $v \in H^1_D(\Omega)$.

Triangulate:



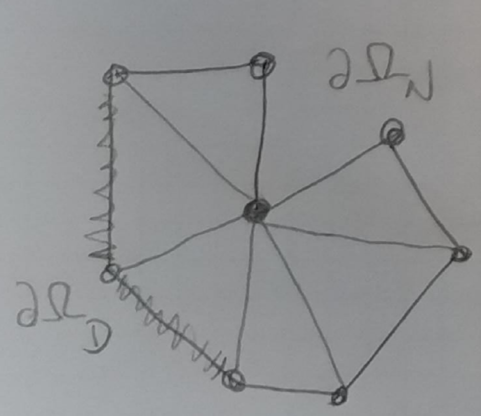
In 1D: Everything is easy! (mod technical rabbish)



In 2D: Things become fun...

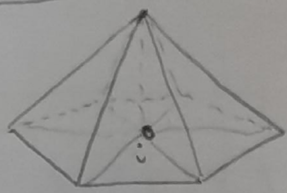
Basic rules to simplify it a bit...

- $\mathcal{T}$ is a finite set of triangles T with $|T|>0$.
- $\bigcup_{T\in\mathcal{T}} T = \overline{\Omega}$
- $T_1 \cap T_2 = \emptyset, e, v$
- $e \in \partial\Omega_D$ xor $e \in \partial\Omega_N$.



In 3D: Same stuff, more fun...

Hat function



$$\Lambda_i = \begin{cases} 1 & \text{at node } i \\ 0 & \text{at other nodes} \end{cases}$$

Discrete weak form

$$\int_\Omega \nabla u_0 \cdot \nabla V \, dx = \int_\Omega f V \, dx + \int_{\partial\Omega_N} g V \, ds - \int_\Omega \nabla u_D \cdot \nabla V \, dx$$

Want to find: $u_0 = \sum_{j=1}^{n} x_j V_j$

and $u = \sum_{j=1}^{n} x_j V_j$ with $x_j = u_D(z_j)$

for z_j on $\partial\Omega_D$.

$$\sum_{k=1}^{n} A_{jk} \underline{x}_k = \underline{b}_j := \text{RHS}.$$

Now,

$$A_{jk} = \int_\Omega \nabla \Lambda_j \cdot \nabla \Lambda_k \, dx = \sum_{T \in \mathcal{T}} \int_T \nabla V_j \cdot \nabla V_k \, dx$$

$j, k = 1, \dots, n.$

Data Structures:

Vertices:

$$V(l,:) = [x_l, y_l] \qquad \# \text{nodes} \times 2$$

Elements:

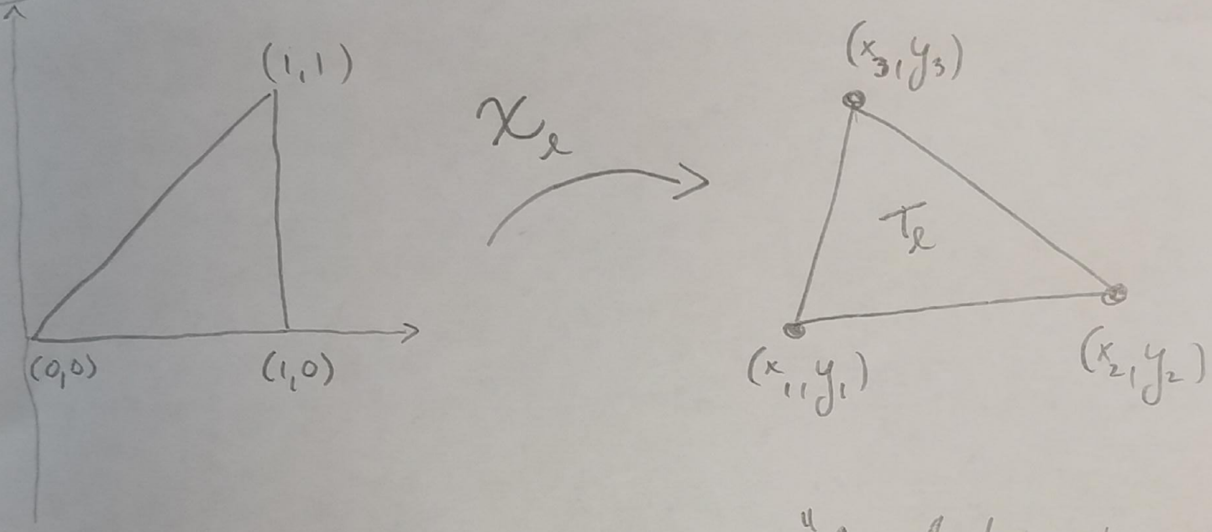
$$E(k,:) = [\underbrace{a \quad b \quad c}_{\text{VERTEX INDEX}}] \qquad \# \text{TRIANGES} \times 3$$

Dirichlet Boundary:

$$\text{dirichlet}(l,:) = [\underbrace{a \quad b}_{\text{VERTEX INDEX}}] \qquad \# \text{B.D.} \times 2$$

BUT, BUT THERE ARE SO MANY TRIANGLES WITH DIFFERENT VERTICES AND SHAPES!!! ...

Solution:



Reference triangle

"Any" triangle

How does $\mathcal{X}_e$ look like?

$$\mathcal{X}_e(\hat{x},\hat{y}) = B_e \begin{pmatrix}\hat{x}\\ \hat{y}\end{pmatrix} + \begin{pmatrix}x_1^e\\ y_1^e\end{pmatrix}$$

$$\begin{pmatrix} x_2^e - x_1^e & x_3^e - x_2^e \\ y_2^e - y_1^e & y_3^e - y_2^e \end{pmatrix}$$