Lecture 4

January 27, 2020

Last week: We calculated some *a posteriori* error estimates

$$\|u-u_h\|_{H^1(\Omega)} \leq C \cdot \underbrace{\text{computable from } u_h}_{\text{see slides for calculation}}$$

$$\implies \frac{1}{\sqrt{C+1}}\|u-u_h\|_{H^1} \leq \sup_{\|v\|_{H^1}=1} R(v) \leq \|u-u_h\|_{H^1}$$

These estimates allow us to create fast adaptive algorithms.

$$\eta(\Delta)^2 = h_\Delta^2 \int_\Delta f^2 + \frac{1}{2}\sum_{E\subseteq\partial\Delta} h_E \int_E \left[\!\left[\frac{\partial u_n}{\partial n}\right]\!\right]^2$$

Here η is the error in Δ. We refine the computation where $\eta(\Delta)$ is "big".

1 Adaptive Algorithms

Adaptive Algorithm:

1. Start from a very coarse discretisation (mesh)
2. Solve the FEM equation $A\vec{c} = \vec{F}$
3. Compute $\eta(\Delta)$ for all Δ
4. Refine those Δ where $\eta(\Delta)$ is "big"
   (a) Mark Δ where $\eta(\Delta)$ is big $\longleftarrow$ *Supply a criterion*
   (b) Refine those Δ $\longleftarrow$ *Supply a refinement procedure*

Adaptive algorithms converge faster than standard uniform mesh refinement.

Question: Why should narrow triangles be avoided?

- Narrow triangles are not good because the long edge dominates the error estimate, and the short edge causes the linear algebra problem to be ill-conditioned, thus difficult to solve. This means our adaptive algorithm must avoid creating long/narrow triangles. This is done a priori.

Question: When is adaptive FEM not a good idea?

- If the solution is smooth, a uniform mesh will be fine. Using the apdaptive algorithm means wasting time calculating unnecessary error estimates.
- If the PDE is inhomogeneous sometimes deriving the error estimates is very difficult or expensive to calculate.

Question: What if the boundary is curved?

- Naıve: Refine the mesh close to $\partial\Omega$

- Use a polynomials for the basis functions rather than a linear basis.

Up Next: Abstract framework for FEM: in arbitrary dimension, for arbitrary (elliptic) PDEs, in a functional analytic framework, repeating what we've seen in 1D. In particular:

$$\int_\Omega (\partial_x u)(\partial_x v) := a(u,v) \underbrace{=}_{\text{FE eq.}} \int_\Omega fv := f(v)$$

In higher dimensions. we will study

$$a(u,v) := \int_\Omega \nabla u \cdot \nabla v = \int_\Omega fv := f(v) \qquad \forall v \in H,$$

corresponding to $-\Delta u = f$.

2 General Setup for Finite Elements

Let H be a Hilbert space i.e. a vector space with a scalar product $\langle u, v\rangle_H$ such that

$$\langle u, v\rangle_H^{\frac{1}{2}} := \|u\|_H,$$

defines a norm and H is complete (important when studying convergence). Think: $H = H_0^1(\Omega)$ the space of functions in which the PDE solution lies. Also let $H_h \subseteq H$ be a finite dimensional subspace – think piecewise linear continuous functions associated to a discretisation of Ω/. In 1D, splitting Ω into intervals, in 2D, splitting in to triangles.

PDE: $a(u,v) = f(v) \quad \forall v \in H$

FE eq. : $a(u_h, v_h) = f(v_h) \quad \forall v_h \in H_h$

2.1 What does this mean?

<u>What is $a(\cdot,\cdot)$?</u>

- It is a <u>bilinear</u> form, $a : H \times H \to \mathbb{R}$ (or $\mathbb{C}$).

$$\begin{aligned} a(\lambda u_1 + \mu u_2, v) &= \lambda a(u_1, v) + \mu a(u_2, v) \\ a(u, \lambda v_1 + \mu v_2) &= \dots \qquad \forall u, v, u_1, u_2, v_1, v_2 \in H, \forall \lambda, \mu \in \mathbb{R} \end{aligned}$$

- It's <u>continuous</u>

$$|a(u,v) \ \leq C\|y\|_H \|v\|_H$$

In 1D:

$$\left|\int_\Omega (\partial_x u)(\partial_x v)\right| \underbrace{\leq}_{\text{C-S}\neq} \left(\int (\partial_x u)^2\right)^{\frac{1}{2}} \left(\int (\partial_x v)^2\right)^{\frac{1}{2}} \leq \|u\|_{H^1} \|v\|_{H^1} \quad \forall u, v \in H$$

- It is coercive:
$$a(u,u) \geq \alpha \|u\|_{H^1}^2 \quad \forall u \in H$$
In 1D:
$$a(u,u) = \int_\Omega (\partial_x u)^2 \underbrace{\geq}_{\text{P-F}\neq} \frac{C}{2}\int_\Omega u^2 + \frac{1}{2}\int_\Omega (\partial_x u)^2 \geq \alpha \|u\|_{H^1},$$
where $\alpha = \min(\frac{1}{2}, \frac{C}{2})$

What is f?
The function f is linear and continuous, $f : H \to \mathbb{R}$.

2.2 How well is the solution to $a(u,v) = f(v)$ approximated by the FE equation?

Examples

1. $$a(u,v) = \int_\Omega (\partial_x u)(\partial_x v) \qquad \text{in } \Omega = [0,1]$$
   - $H = H_0^1(\Omega), H_h$ piecewise linear continuous functions
   - Continuity $\longleftarrow$ C-S inequality
   - Coercivity $\longleftarrow$ P-F inequality
   - Continuity of f
   $$\left|\int_\Omega fv\right| \leq \left(\int_\Omega f^2\right)^{\frac{1}{2}} \left(\int_\Omega v^2\right)^{\frac{1}{2}} = \|f\|_{L^2}\|v\|_{L^2}$$
   $$\leq \|f\|_{L^2}\|v\|_{H^1} = C\|v\|_{H^1}$$
   So continuous if $f \in L^2$ (or if $f \in H^{-1}(\Omega)$).

2. In higher dimensions,
$$a(u,v) := \int_\Omega \nabla u \cdot \nabla v = \int_\Omega fv := f(v) \qquad \forall v \in H,$$
where
$$H = H_0^1(\Omega) = \{v \in L^2(\Omega), \partial_{x_j} v \in L^2(\Omega)\, \forall j\}.$$
Members of this set are called *"functions of finite energy"*, and the energy norm is
$$\|u\|_H = \left[\|u\|_{L^2(\Omega)}^2 + \sum_{j=1}^n \|\partial_{x_j} u\|_{L^2(\Omega)}^2\right]^{\frac{1}{2}}$$
H_h is any subspace of$H$ of finite dimension. The choice of subspace is motivated by knowledge of the solution or ease of implementation. For example if you're solving a wave equation, you may choose $\cos$ and $\sin$ as basis functions; if the solution has kinks a linear functions will be better; or if the solution is smooth, you could use high order polynomials. Usually we do not think of H_h as fixed but consider a family $\{H_h\}_{h \in I \subseteq (0,\infty)}$ of subspaces of H such that $H_{h_1} \supseteq H_{h_2}$ whenever $h_1 \leq h_2$.
$$\bigcup_{h \in I} H_h \text{ is dense in } H$$
For the theory, think of spaces of functions, not of meshes/discretisations.

- Continuity

$$\|a(u,v)\| = \left|\int_\Omega \nabla u \cdot \nabla v\right| \leq \left(\int_\Omega (\nabla u)^2\right)^{\frac{1}{2}} \left(\int_\Omega (\nabla v)^2\right)^{\frac{1}{2}} \leq \|u\|_{H^1}\|v\|_{H^1}$$

- Coercivity

$$a(u,u) = \int_\Omega (\nabla u)^2 \geq \alpha \|u\|_{H^1}^2$$

  which follows from a higher dimensional P-F inequality.
- Continuity of f follows from C-S inequality.

Does the bilinear form above actually still relate to a PDE?

Theorem 1. *$u \in H_0^1(\Omega)$ solves $a(u,v) = f(v)\, \forall v \in H_0^1(\Omega)$ if and only if*

$$\begin{aligned} -\Delta u &= f \text{ in } \Omega \\ u &= 0 \text{ on } \partial\Omega \end{aligned}$$

Proof. Identical to the 1D case, using Green's theorem. □

This is not restricted to the diffusion equation, we could add more terms. Let $A \in \mathbb{R}^{n\times n}$ be positive definite. Then the theorem becomes

$$\begin{aligned} a(u,v) = &\int_\Omega (A(x)\nabla u)\cdot \nabla v + b(x)uv + \vec{c}\,(\nabla u)v = f(v)\, \forall v \in H_0^1(\Omega) \\ \iff &-\nabla(A(x)\nabla u) + \vec{c}\,\nabla u + bu = f \text{ in } \Omega \\ &u = 0 \text{ on } \partial\Omega \end{aligned}$$

If $a(\cdot,\cdot)$ is coercive, this places large restrictions on $b(x), c$. However we actually only require the highest order part of the operator to be coercive. For now, we assume all of $a(\cdot,\cdot)$ is coercive. Could extend theory to

$$a(u,v) = \underbrace{a_1(v,u)}_{\text{coercive}} + \underbrace{a_2(u,v)}_{\text{compact}}.$$

In this abstract setting, one obtains

Theorem 2 (Cea's Lemma)**.** *Let $u \in H$ be the solution to $a(u,v) = f(v)\, \forall v \in H$ and $u_h \in H_h$ be the solution to $a(u_h, v_h) = f(v_h)\, \forall v_h \in H_h$. Then*

$$\|u - u_h\|_H \leq C \inf_{v_h \in H_h} \|u - v_h\|_H.$$

That is, the FE error is less than a constant times the error of the best possible approximation.

If $\bigcup H_h$ is dense in H, the right hand side tends to 0 as $h \to 0$ and the method converges.

Next Week: Implementation of FEM

After that: Lax-Milgram and proof of Cea's Lemma