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Numerical Analysis of PDEs Lecture 3

Recap:

The model equation:

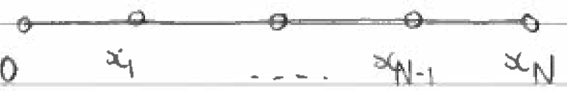
$$-\partial_x^2 u = f \qquad \text{in } \quad \Omega = (0,1) \qquad (1)$$

$$u(0) = u(1) = 0$$

Two approaches discussed so far:
- Finite Differences
- Finite Elements

Finite Differences:

- Interval split into N intervals of length $h = \frac{1}{N}$



- For $j \in \{1, \dots, N-1\}$, difference quotients are used to ~~a~~ approximate at each x_j ~~with~~ with

$$\frac{-u_{j+1} + 2u_j - u_{j-1}}{h^2} = f(x_j)$$

This leads to the matrix equation

$$A\vec{u} = \vec{F}$$

which can be solved for

$$\vec{u} = (u_0, \dots, u_N)^T$$

Finite Elements:

- Similarly to Finite Difference, Finite Elements produces a matrix equation:

$$A\vec{u} = \vec{F} \qquad (2)$$

which is to be solved to obtain the approximate solution u_h.

- However, unlike for finite differences the matrix equation is arrived at only after the model equation (1) has been converted into the equivalent minimisation problem:

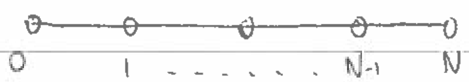
Find $v_h \in H_h \subset H_0^1(\Omega)$ such that

$$E(v_h) = \frac{1}{2}\int_\Omega (\partial_x v_h)^2 dx - \int_\Omega f v_h dx$$

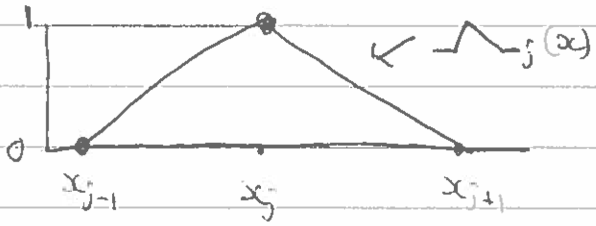
is minimised over H_h.

- Here $H_0^1(\Omega) = \{v \in L_2(\Omega) : \partial_x v \in L_2(\Omega) \text{ and } v(0) = v(1) = 0\}$

- The subspace $H_h \subseteq H_0^1(\Omega)$ is defined by discretising the unit interval into N subintervals



- Piecewise Linear hat functions are then defined as Λ_j such that for $j \in \{1, \dots, N-1\}$



- The N-1 hat functions (Λ_j) then ~~form~~ span the space |'

- The finite dimensional subspace $H_h \subseteq H^1_0$ is then defined as the span of $\{\Lambda_j\}_{j=1}^{N-1}$.

▯ The weak formulation of (1) is: Find $u \in H^1_0(\Omega)$ s.t.

$$\int_\Omega \partial_x u \, \partial_x v = \int_\Omega f v \qquad \forall\, v \in H^1_0(\Omega)$$

The space $H^1_0(\Omega)$ may also be referred to as "Functions of finite energy"

- Finite Element formulation of (1):
  Find $u_h \in H_h$ s.t

$$\int_\Omega \partial_x u_h \partial_x v_h \, dx = \int_\Omega f v_h \qquad \forall \, v_h \in H_h$$

Which leads to the algebraic system

$$A\vec{u} = \vec{F}$$

Poincaré-Friedrichs inequality:

$$\int_\Omega v^2 \leq C_1 \int_\Omega (\partial_x v)^2 \qquad \forall v \in H_0^1(\Omega)$$

i.e.

$$\|v\|_{L^2} \leq C_1^{1/2} \|\partial_x v\|_{L^2}$$

- This result shows that the matrix A in (2) is positive definite and hence invertible.

Both the weak formulation and the finite element formulation are gradient tests.

Last lecture it was shown that if u is the solution of the weak formulation and u_h the solution to the finite element formulation then:

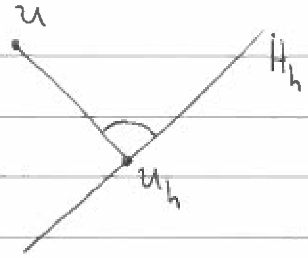
$$\int_\Omega \partial_x(u-u_h)\partial_x v_h = 0 \qquad \forall v_h \in H_h$$

This result was labelled "Galerkin Orthogonality".

For the scalar product $\int_\Omega \partial_x(\cdot)\partial_x(\cdot)$ the "error" $u-u_h$ is orthogonal to the space H_h so u_h is the best approximation to u:

$$\int\left[\partial_x(u-u_h)\right]^2 dx = \inf_{v_h \in H_h}\left(\int \partial_x(u-v_h)\right)^2$$

~~and~~



Cea's Lemma:

There exists $C_2 > 0$, independent of H_h so that

$$\|u-u_h\|_{H^1_0(\Omega)} \le C_2 \inf_{v_h \in H_h} \|u-v_h\|_{H^1_0(\Omega)}.$$

This week (week 3)

- Introduce "a priori" estimates in $\|\cdot\|_{H^1}$, $\|\cdot\|_{L^2}$ and $\|\cdot\|_{L^\infty}$
- Introduce "a posteriori" estimates and adaptive finite elements.

"a priori" estimates

It will be shown how $\|u-u_h\|_{H^1}$, $\|u-u_h\|_{L^\infty}$ and $\|u-u_h\|_{L_2}$ change with level of refinement, h.

- $$\|u-u_h\|_{H^1}^2 = \int_\Omega (u-u_h)^2\,dx + \int_\Omega \left[\partial_x(u-u_h)\right]^2 dx$$
- $$\|u-u_h\|_{L_2}^2 = \int_\Omega (u-u_h)^2\,dx$$
- $$\|u-u_h\|_{L_\infty} = \sup_{x\in\Omega} |u(x)-u_h(x)|$$

Firstly, the result from last lecture

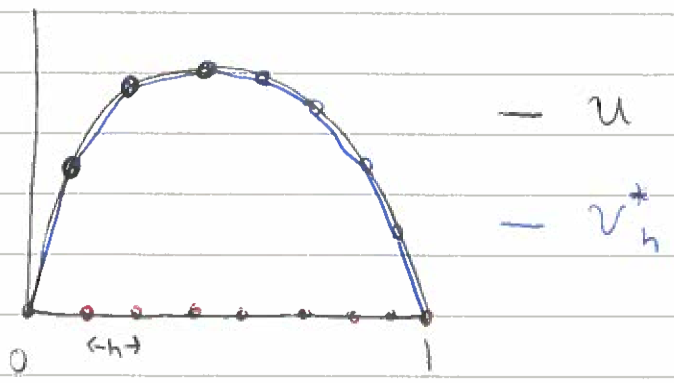
$$\left[\inf_{v_h \in H_h} \|u - v_h\| \to 0 \quad \text{as } h \to 0 \right]$$

will be shown.

Proof:

Let $v_h^* \in H_h$ be such that for each node x_j, of the discretisation $v_h^*(x_j) = u(x_j)$ and v_h^* is the interpolation of the points $v_h^*(x_j)$.

I.E.



for the interval $[x_j, x_{j+1}]$

By the Poincaré-Friedrichs inequality it is then the case that

$$\int_{\Omega}^{1} (u - v_h^*)^2 \, dx = \| u - v_h^* \|^2_{L_2(\Omega)}$$

$$= \sum_{j=1} \int_{x_j}^{x_{j+1}} (u - v_h^*)^2 \, dx$$

$$\leq C_1 h^2 \sum_{j=1} \int_{x_j}^{x_{j+1}} (\partial_x (u - v_h^*))^2 \, dx$$

$$= C_1 h^2 \int_0^1 (\partial_x (u - v_h^*))^2 \, dx \qquad (3)$$

By the same arguments it is the case that

$$\int_0^1 (\partial_x (u - v_h^*))^2 \, dx$$

$$\leq C_1 h^2 \sum_j \int_{x_j}^{x_{j+1}} (\partial_x^2 u - \partial_x^2 v_h^*)^2 \, dx$$

Since $v_h^* \in H_h$ it is the case that $\partial_x^2 v_h^* = 0$.

(v_h^* is linear)

it now follows that

$$\int_0^1 \left(\partial_x (u - v_h^*)\right)^2 dx$$

$$\leq C_1 h^2 \int_0^1 (\partial_x^2 u)^2 \, dx$$

$$\leq \tilde{C}_1 h^2$$ where $\tilde{C}$ has absorbed $\int_0^1 (\partial_x^2 u)^2 dx$.

(4)

From results (3), (4) and by the definition of the H^1 norm it follows that

$$\|u - v_h^*\|_{H^1}^2 = \int_0^1 (u - v_h^*)^2 dx + \int_0^1 \left(\partial_x (u - v_h^*)\right)^2 dx$$

$$\leq (C_1 h^2 + 1) \int_0^1 \left(\partial_x (u - v_h^*)\right)^2 dx$$

$$\leq (1 + C_1 h^2) \tilde{C}_1 h^2$$

$$\leq \tilde{\tilde{C}}_1 h^2$$

Thus showing that

$$\inf_{v_h \in H_h} \|u - v_h\|_{H^1}$$

$$\leq \|u - v_h^*\|_{H^1}$$

$$\leq \sqrt{\tilde{\tilde{c}}_1}\, h$$

Therefore, by Cea's Lemma, it is the case that

Corollary 1

$$\|u - u_h\|_{H^1} \leq C_2 \inf_{v_h \in H_h} \|u - v_h\|_{H^1} \leq C_2 \sqrt{\tilde{\tilde{c}}_1}\, h = C_3 h$$

Corollary 2

$$\|u - u_h\|_{L^2} \leq C_3 h$$

Proof

$$\|u - u_h\|_{L_2} \leq \left(\|u - u_h\|_{L_2}^2 + \|\partial_x (u - u_h)\|_{L_2}^2 \right)^{\frac{1}{2}}$$

$$= \|u - u_h\|_{H^1} \leq C_3 h$$

Corollary 1 is the "a priori" estimate of the H^1 ~~norm~~ error and shows that the H^1 error is linear with the level of refinement h.

Corollary 2 shows that the L_2 error also decreases linearly with increasing refinement.

(refinement increase $\Rightarrow$ h decrease).

it will now be shown that ~~one can~~ in the case of $\|u-u_h\|_{L_2}$, one can do better than linear convergence.

Theorem: (Aubin, Nitsche)

$\exists\, C_4 > 0$ such that

$$\|u-u_h\|_{L_2} \leq C_4 h^2$$

In other words the convergence in L_2 (up to a constant) is twice as fast as that in H^1.

Proof (Aubin, Nitsche)

Consider the auxillary problem

$$\begin{cases} -\partial_x^2 w = u - u_h \\ w(0) = w(1) = 0 \end{cases} \qquad (5)$$

It is the case that

$$\|u - u_h\|_{L_2}^2$$

$$= \int_0^1 (u-u_h)(u-u_h)\,dx$$

$$= \int_0^1 (u-u_h)(-\partial_x^2 w)\,dx \qquad \text{from (5)}$$

$\uparrow$

Integration by parts

$\downarrow$

$$= \int_0^1 \partial_x(u-u_h)(\partial_x w)\,dx$$

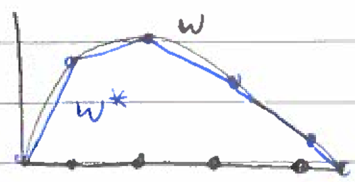
$\uparrow$ By Galerkin orthogonality

$\downarrow$

$$= \int_0^1 \partial_x(u-u_h)\,\partial_x(w - v_h)\,dx \qquad \forall\, v_h \in H_h$$

Now specifically pick $v_h = w^*$ where w^* interpolates w at the nodes.

I.E



It is now the case that

$$\|u-u_h\|^2_{L_2}$$

$$= \int_0^1 \partial_x(u-u_h)\,\partial_x(w-w^*)\,dx$$

By Cauchy-Schwarz

$$\leq \|\partial_x(u-u_h)\|_{L_2} \|\partial_x(w-w^*)\|_{L_2}$$

. From Corollary 1 it is the case that

$$\|\partial_x(u-u_h)\|_{H^1} \leq C_3 h \qquad (6)$$

. From the proof of result (4) it is also the case that

$$\|\partial_x(w-w^*)\|_{L_2}$$

$$\leq \sqrt{C_1}\, h \,\|\partial_x^2 w\|_{L_2} \qquad (7)$$

Results (6) and (7) now combine to show

$$\|u - u_h\|_{L_2}^2$$

$$\le C_3\sqrt{C_1}\, h^2 \, \|\partial_x^2 w\|_{L_2}$$ By definition of auxillary problem (5)

$$= C_3\sqrt{C_1}\, h^2 \, \|u - u_h\|_{L_2}$$

$$= C_4 h^2 \|u - u_h\|_{L_2}$$

Therefore

$$\|u - u_h\|_{L_2} \le C_4 h^2 \qquad \square$$

<u>Behavior of $\|u - u_h\|_{L_\infty}$ with h.</u>

- In 1 dimension (as a result of Sobolev Embedding)

$$\|u - u_h\|_{L^\infty} = \sup_{x \in \Omega} |u(x) - u_h(x)| \le C_5 \|u - u_h\|_{H^1}$$

$$\le C_5 C_3 h$$

$$= C_6 h$$

- In 2 dimensions

$$\|u - u_h\|_{L_\infty} \le C_7 h \log(h).$$

Upshot:

- There are different ways of measuring errors and all converge at their own rates as h decreases.
- FEM converges, but rate of convergence depends on which norm is used to measure the error.
- All error estimates covered so far are

  "a priori".

"A Posteriori Estimates"

If it is the case that the rhs, f, of (1) has been ~~a priori~~ prescribed then, given a computed FEM solution $u_h \in H_h$, more can be said about the bounds on the error measurements. Thus motivating the use of adaptive finite elements.

Simple example: By Poincaré Friedrichs it is the case that $\exists\ C_g > 0$ such that

$$\frac{1}{C_g} \| u - u_h \|_{H^1}^2 \leq \int_0^1 \left[\partial_x (u - u_h) \right]^2$$

Carrying on, it can be seen that

$$\frac{1}{C^8}\left\|u-u_h\right\|_{H^1}^2$$

$$\leq \int_0^1 \partial_x(u-u_h)\,\partial_x(u-u_h)\,dx$$

By Galerkin orthogonality

$$= \int_0^1 \partial_x(u-u_h)\,\partial_x(u-v_h)\,dx \qquad (\text{For any } v_h \in H_h)$$

$$= \sum_j \int_{x_j}^{x_{j+1}} \partial_x(u-u_h)\,\partial_x(u-v_h) \qquad \text{where } v_h \in H_h$$

$$\cancel{= \sum_j \int_{x_j}^{x_{j+1}} \partial_x(u-u_h)}$$

$$= \sum_j \left(\int_{x_j}^{x_{j+1}} \partial_x u\,\partial_x(u-v_h) - \int_{x_j}^{x_{j+1}} \partial_x u_h\,\partial_x(u-v_h) \right) \qquad (8)$$

By the definition of the weak formulation of the problem

$$\int_{x_j}^{x_{j+1}} \partial_x u \, \partial_x (u - v_h) = \int_{x_j}^{x_{j+1}} f(u - v_h)$$

$$\Rightarrow \sum_j \int_{x_j}^{x_{j+1}} \partial_x u \, \partial_x (u - v_h) = \int_0^1 f(u - v_h) \qquad (9)$$

Furthermore from integration by parts it follows that

$$\sum_j \int_{x_j}^{x_{j+1}} \partial_x u_h \, \partial_x (u - v_h)$$

$$= \sum_j -\int_{x_j}^{x_{j+1}} \partial_x^2 u_h (u - v_h) + \left[\partial_x u_h (u - v_h) \right]_{x_j}^{x_{j+1}}$$

u_h is linear

$$= 0 + \left[\partial_x u_h (u - v_h) \right]_{x_j}^{x_{j+1}} \qquad (10)$$

This then means that by (8), (9) and (10)

$$\frac{1}{C^8}\|u-u_h\|^2_{H_1}$$

$$\leq \|f\|_{L_2}\|u-V_h\|_{L_2} + \sum_j (\text{jump of } \partial_x u_h \text{ at } x_j)\|u-V_h\|_{L^\infty}$$

We are free to pick $V_h = u^* \in \mathbb{V}_h$ where u^* is the linear interpolant of u.

$$= \|f\|_{L_2}\|u-u^*\|_{L_2} + \sum_j (\text{jump of } \partial_x u_h \text{ at } x_j)\|u-u^*\|_{L^\infty}$$

Now by the results established in the proof of corollary 1 $\exists$ C_9 such that

$$\|u-u^*\|_{L_2} \leq C_9 h \qquad (11)$$

Furthermore by the a priori estimates for $\|u-u_h\|_{L^\infty}$ $\exists$ C_{10} s.t

$$\|u-u^*\|_{L^\infty} \leq C_{10} h \qquad (12)$$

Results (11) and (12) then combine to show that

$$\|u-u_h\|_{H^1}^2$$

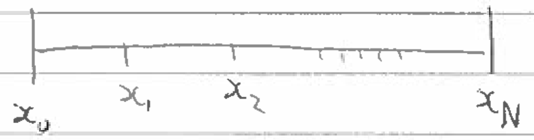
$$\leq C_{11} \sum_j h\left(\|f\|_{L_2(x_j, x_{j+1})} + (\text{jump of } \partial_x u_h \text{ at } x_j)\right) \qquad (13)$$

where $C_{11} = \max(C_8 C_9, C_8 C_{10})$.

The rhs of (13) is now entirely computable hence one can control the error of the approximate solution by the size of h and the jump of $\partial_x u_h$ at x_j.

This introduces the idea of adaptive algorithms, described now in the following steps

⓪ Given an initial partition J_0 of Ω



$$J_0 = \{[x_0, x_1], [x_1, x_2], \ldots, [x_{N-1}, x_N]\}$$

① Solve FEM Equation for this partition J_0 for u_h

$$A\vec{u} = \vec{F}$$

(2) Now compute

$$\eta_j = h\|f\|_{L_2[x_j, x_{j+1}]} + \frac{1}{2}h(\text{jump of } \partial_x u_h \text{ at } x_j)$$

$$+ \frac{1}{2}h(\text{jump of } \partial_x u_h \text{ at } x_{j+1})$$

(3) If η_j is "big", then divide $[x_j, x_{j+1}]$ into smaller intervals.

For a desired $\varepsilon > 0$, once steps (0) → (3) have been carried out check if

$$\left(C_{11} \sum h\|f\|_{L_2[x_j, x_{j+1}]} + h(\text{jump of } \partial_x u_h \text{ at } x_j)\right) < \varepsilon$$

If the condition is satisfied then stop procedure (0) → (3).

[Ways of defining "big" in point 3]

Definition 1: Maximal Marking

Given some $\delta \in (0,1)$, η_j is big if

$$\eta_j > \delta \max_I \eta_I$$

Definition 2: Dörfler Marking

- Order η_j's according to size

e.g.

$$\eta_1 = 0.7$$

$$\eta_2 = 0.3$$

$$\eta_3 = 0.1$$

$$\eta_4 = 0.03$$

- Total sum $\sum_j \eta_j = 1.13$.

- Choose some threshold θ such that $0 < \theta \leq 1.13$

- Sum η_j from 1 until $\sum_{j=1}^{M} \eta_j > \theta$

- Refine intervals $[x_j, x_{j+1}]$ up to $[x_M, x_{M+1}]$.

Divide each interval in two.