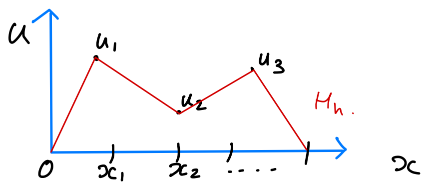
Lecture 2

14/1/20.

$$* \begin{cases} -\partial_x^2 u = f & \text{in } \Omega = (0,1) \\ u(0) = u(1) = 0 \end{cases}$$



$x_j = jh$

Basic strategy in any disc. method: It tries to convert the PDE to algebra, i.e.

$-\partial_x^2 \to \underline{A}$ $\quad * \ \underline{A}u = \underline{F}$

Last week we saw FD and FEM.

- FD: $-\partial_x^2 u(x) = \dfrac{2u(x) - u(x-h) - u(x+h)}{h^2}$

$$= \frac{2u_j - u^{j-1} - u^{j+1}}{h^2}. \quad \text{Then } \underline{A} = \begin{pmatrix} 2 & -1 & & 0 \\ -1 & 2 & -1 & \\ & -1 & \ddots & -1 \\ 0 & & -1 & 2 \end{pmatrix}$$

- FEM: Leads to the 'same' linear Algebra problem but from a different point of view

Basic Idea

u is a sol^n to $*$ iff u is a minimiser of $E(v) = \frac{1}{2}\int_0^1 (\partial_x v)^2 dx - \int_0^1 fv\, dx$ over $H = \{v : (0,1) \to \mathbb{R} : \int_0^1 v^2 < \infty,\ \int_0^1 (\partial_x v)^2 < \infty,\ v(0) = v(1) = 0\} = H_0^1(\Omega)$ (Sobolev space with 0 BC).

Then the above Th^m leads to a numerical method for $*$: We solve $*$ not by evaluating the second order PDE but rather we minimise E over finite dimensional subspace $H_h \subseteq H$.

Note: $H_h = \{v_h : (0,1) \to \mathbb{R}$ linear and continuous in $[x_j, x_{j+1}]$ $\forall j$ $v_h(0) = v_h(1) = 0\}$

$= \text{Span}\{\Lambda_j : j = 1, \ldots, N\}$, Λ_j = p.w. linear.

$$\Lambda_j(x_i) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

Inserting this into E gives, $\quad H_h \ni v_h = \sum_{j=1}^{N} c_j \Lambda_j \Rightarrow E(v_h) = E(\sum_{j=1}^{N} c_j \Lambda_j) = \frac{1}{2}\underline{c}^T \underline{A}\, \underline{c} - \underline{F}\, \underline{c}$

where $\underline{A}_{ij} = \int_0^1 (\partial_x \Lambda_i)(\partial_x \Lambda_j)\, dx$.

$\therefore$ We have converted $*$ into a quadratic minimisation problem:

$$\text{Minimise } \tfrac{1}{2}\underline{c}^T \underline{A}\, \underline{c} - \underline{F}\, \underline{c}$$

Over $\underline{c} \in \mathbb{R}^N$. Then applying the Gaussian test

$\frac{\partial}{\partial c_j}(\ldots) \overset{!}{=} 0 \quad \forall j \iff \underline{A}\, \underline{c} = \underline{F}$, from LA.

Note: There is no reason to expect this to work if your f^n is C^2.

Th^m 1

Assume u is sufficiently smooth. Then u is a sol^n to $*$ iff u minimises the 'energy functional'

$$E(u) = \frac{1}{2}\int_0^1 (\partial_x v)^2\,dx - \int_0^1 f v\,dx$$

among all v with $\int v^2,\ \int(\partial_x v)^2 < \infty,\quad v(0) = v(1) = 0$ - sufficiently smooth.

Using hölders or C-S inequality gives the above.

Proof of Thm from L1.

The basic computation $E(u+h) - E(u)\quad \forall\ h \in H$.

Then,

$$E(u+h) - E(u) = \int_0^1 (\partial_x(u+h))^2 - \int_0^1 f(u+h) \quad - \frac{1}{2}\int_0^1 (\partial_x u)^2 + \int_0^1 fu$$

$$= \int_0^1 \partial_x u\, \partial_x h \ + \frac{1}{2}\int_0^1 (\partial_x h)^2 - \int_0^1 f h.$$

$$= -\int_0^1 (-\partial_x^2 u - f)h \ + \cancel{[(\partial_x u)h]_{0=x}^{1}} \quad + \frac{1}{2}\int_0^1 (\partial_x u)^2$$

noting $h \in H \Rightarrow h(0) = h(1) = 0$.

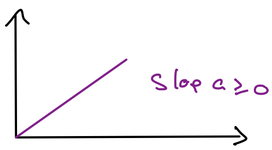
"$\Rightarrow$"

Assume that u solves $(*)$. This implies $E(u+h) - E(u) = \frac{1}{2}\int_0^1 (\partial_x h)^2 \geq 0 \quad \forall\ h \in H$ which yields that u minimises E over H, so we also see that this is a unique minimum.

"$\Leftarrow$"

Assume u is a minimiser of E over H: $0 \leq E(u+\lambda h) - E(u) \quad \forall\ h \in H$

$$0 \leq E(u+h) - E(u) = \lambda\int_0^1 (-\partial_x^2 u - f)h \ + \frac{\lambda^2}{2}\int_0^1 (\partial_x h)^2$$
$$= a\lambda + b\lambda^2 \geq 0$$



Slope $a \geq 0 \Rightarrow a = 0$

$$\Rightarrow \quad a = 0 = \int_0^1 (-\partial_x^2 u - f)h \qquad \forall\ h \in H.$$

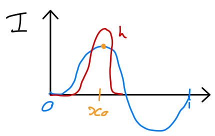
Now there are many ways to argue the above. The Fundamental Th^m of the Calculus of variations gives:

$$\int F h = 0 \quad \forall\ h \in H \rightarrow F = 0$$

~~Two proofs~~

1) Functional Analysis: Hahn-Banach th^m: If $\langle w, h\rangle = 0 \quad \forall\ h \in H \Rightarrow w = 0$

2) Assume ~~that~~ $I = -\partial_x^2 u - f$ is continuous, sufficiently smooth, and non-zero:



We see ~~that~~ $0 = \int h(-\partial_x^2 u - f) \geq 0$, which yields a contradiction.

It's useful to know how the methods are built and not just considered as a LA problem.

Showing that FEM works

Fact 1

The matrix $\underline{A}$ must be invertible and positive definite. This means that there exists a unique Sol^n to the discretised problem.

Proof

Note that $\underline{A}$ is a square matrix so we want to show $\ker \underline{A} = \{0\}$.

This follows if we show that $0 = \underline{c}^T \underline{A} \underline{c} \Rightarrow \underline{c} = 0$.

$$\underline{c}^T \underline{A}\, \underline{c} = \sum_{ij} c_i^T \underline{A}_{ij} c_j \qquad (c_i^T = c_i)$$

$$= \sum_{ij} c_i c_j \int \partial_x \Lambda_i \, \partial_x \Lambda_j = \sum_{ij} \int (c_i \partial_x \Lambda_i)(c_j \partial_x \Lambda_j)$$

$$= \int \Big(\sum_j c_j \partial_x \Lambda_j\Big)\Big(\sum_j c_j \partial_x \Lambda_j\Big)$$

$$= \int \Big(\partial_x \sum_i c_j \Lambda_j\Big)^2 = \int (\partial_x v_h)^2 \geq 0 \qquad \text{Unless } v_h \equiv 0 \Leftrightarrow \underline{c} = 0.$$

where $v_h = \sum c_j \Lambda_j \in H$

'Fun' Fact

This also shows that the FD problem has a unique Sol^n.

Fact 2

The Sol^n $u_h \in H_h$ of the discretised minimisation problem for E (i.e. the FEM Sol^n u_h) is quasi-optimal in the following sense:

Consider the L^2 norm of

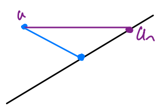
$$e = \|\overset{\text{Exact}}{u} - \overset{\text{FEM}}{u_h}\|^2_{L^2(0,1)} + \|\partial_x u - \partial_x u_h\|^2_{L^2(0,1)} \leq \overset{\text{Geometry, length of interval + Laplace operator}}{G}\big(\|u - v_h\|^2_{L^2(0,1)} + \|\partial_x u - \partial_x v_h\|^2_{L^2}\big)$$

$$\forall\ v_h \in H_h$$

$\Rightarrow e = \|u - u_h\|^2_{H^1(0,1)}$ where the superscript 1 corresponds to a first order derivative in the above summation.

Cea's Lemma

i.e. $\|u - u_h\|_{H^1} \leq G \|u - v_h\|_{H^1(0,1)} \quad \forall\ v_h \in H_h$.



best possible approximation

$\Rightarrow$ How well we can approximate u by f^ns in H_h.

Note: If $G = 1$ then this is optimal otherwise it is quasi-optimal

Lemma (Poincaré-Friedrichs inequality)

$$\int_0^1 w^2 \, dx \leq \int_0^1 (\partial_x w)^2 \, dx \qquad \forall\ w \in H_0^1(0,1)$$

Proof

Suffices to show this for $w \in C_0^1(0,1) \subseteq H_0^1(0,1)$ a dense subspace.

Now using the FTC:

$$\int_0^1 \left| \begin{array}{l} w(x) = w(0) + \int_0^x \partial_x w = \int_0^x \partial_x w \\ \Rightarrow w^2(x) = \left(\int_0^x 1 \, \partial_x w\right)^2 \underset{C-S}{\leq} \left(\int_0^x 1^2\right)^{\frac{2}{2}} \left(\int_0^x (\partial_x w)^2\right)^{\frac{2}{2}} \leq \left(\underbrace{\int_0^1 1}_{=1}\right)\left(\int_0^1 \partial_x w\right) \end{array} \right)$$

Using P-F we can prove Cea's lemma:

Proof (Cea's Lemma)

1) Apply P-F to $w = u - u_h$

$$\int (\partial_x (u-u_h))^2 + \int (u-u_h)^2 \leq 2 \int (\partial_x (u-u_h))^2$$

$$\Leftrightarrow \quad \|u-u_h\|_{H^1}^2 \leq 2 \int (\partial_x (u-u_h))^2$$

2) FEM problem revisited

Minimise $E(v_h) = \frac{1}{2} \int (\partial_x v_h)^2 - \int f v_h$ over $v_h \in H_h$ where u_h is the minimiser.

Eq^n satisfied by the minimiser u_h: in the proof of the Th^m "$a = \int (-\partial_x^2 u - f) h = 0$"
We show the above again for u_h,

$$\overset{u_h \text{ minimised}}{0 \leq} E(u_h + \lambda v_h) - E(u_h) \quad \forall\ v_h \in H_h \quad \forall\ \lambda \in \mathbb{R}$$
$$= \lambda \int_0^1 (\partial_x u_h)(\partial_x v_h) + \frac{\lambda^2}{2} \int (\partial_x v_h)^2 - \lambda \int_0^1 f v_h$$

As before this is greater than zero for all λ implies that the coefficient of λ' needs to be 0

$$\int_0^1 (\partial_x u_h)(\partial_x v_h) = \int_0^1 f v_h \quad \forall\ v_h \in H_h \quad \text{FEM Eq}^n.$$
$$\int_0^1 (\partial_x u)(\partial_x v) = \int f v \quad \forall\ v \in H_0^1(0,1) \quad \text{Eq}^n \text{ satisfied by } u.$$

3 Galerkin Orthogonality

Use the second of these eq^{n}s with $v = v_h \in H_h$.

$$(i) \quad \int_0^1 (\partial_x u)(\partial_x v_h) = \int_0^1 f v_h$$
$$(ii) \quad \int_0^1 (\partial_x u_h)(\partial_x v_h) = \int_0^1 f v_h \quad \text{FEM Eq}^n$$

(i) - (ii) yields : $\int_0^1 [\partial_x (u-u_h)] \partial_x v_h = 0$

Showing that the error is orthogonal to v_h in this scalar product but not in the H^1 scalar product $\langle u, v \rangle_{H^1} = \int uv + \int \partial_x u\, \partial_x v$

4 Concluding the proof

Note addition of: $v_h - v_h$

$$\frac{1}{2} \|u-u_h\|_{H^1}^2 \leq \int \partial_x (u-u_h) \partial_x (u - v_h) + \underbrace{\int \partial_x (u-u_h)\, \partial_x (v_h - u_h)}_{=0} \quad \text{from step 3.}$$

$$\overset{C\text{-}S}{\leq} \left(\int (\partial_x (u-u_h))^2 \right)^{\frac{1}{2}} \left(\int (\partial_x (u-v_h))^2 \right)^{\frac{1}{2}}$$

$$\leq \|u-u_h\|_{H^1} \|u-v_h\|_{H^1}$$

$$\leq \|u-u_h\|_{H^1} \|u-v_h\|_{H^1}$$

$$\Rightarrow \|u-u_h\| \leq 2 \|u-v_h\|_{H^1} \quad \forall\ v_h \in H_h$$ Note we cancel the square.

$\blacksquare$

To Do

1 General Framework: Elliptic PDEs in $H_0^1(\Omega)$ where $\Omega \subseteq \mathbb{R}^d$ bounded domain. (Coercive bilinear forms on Hilbert spaces.

2 Show that finite elements works:

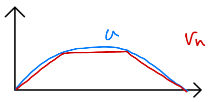
i.e. Show Ceas lemma Converges $\|u-v_h\|_{H^1} \xrightarrow{h\to 0} 0$. This is from approximation theory: How well can I approximate the unknown sol^n u by functions $v_h \in H_h$?

3 Advanced Algorithms

Adaptive Meshing; fast Solvers; Spectral FEM.

Recall

We are trying to approximate the unknown sol^n u by p.w. linear f^{n}s:



How good is this approximation as the mesh size $h \to 0$? Does the RHS of Cea's lemma go to zero?

Basic Observation

P-F inequality on the interval $I_j = [x_j, x_{j+1}]$:

If $w \in H_0^1(I_j)$ then $\int_{I_j} w^2 \leq C h^2 \int_{I_j} (\partial_x w)^2$, i.e. the constant for P-F should behave like h.

Exercise

Go through the proof of P-F with interval of length h instead of 1.

$\Rightarrow \int_0^1 (\underbrace{u-u_h}_{=w})^2 = \sum_{i=1}^N \int_{I_j} (u-v_h)^2 \underset{P-F}{\leq} C h^2 \sum_i \int (\partial_x(u-v_h))^2$ by choosing v_h to be $v_h(x_j) = u(x_j)$ and linear

linearly interpolated in between $\forall j$.

$$\int_0^1 (\underbrace{u-u_h}_{=w})^2 = \sum_{i=1}^N \int_{I_j} (u-v_h)^2 \underset{P-F}{\leq} C h^2 \sum_i \int (\partial_x(u-v_h))^2$$

$$\Leftrightarrow \|u-v_h\|_{L^2}^2 \leq C h^2 \|u-v_h\|_{H^1}$$