27/01/20

Numerical Analysis of PDEs

Refs. Quarteroni, Sacco, Saleri, Numerical mathematics : - good for a broad overview.
- bad for PDE.

Lecture 1.

Some contents of this course

- Solving PDE in complex domains

How?
- Finite element methods
  - → engineering applications
  - → especially static and heat problems
- Finite differences
  - → simple problems and hyperbolic / wave problems
- Time stepping methods:
  ~~combining~~ combine methods for spatial problems to solve time-dependent problems.
- Numerical linear algebra.
  (... such as spectral methods etc.)

Does it work?
- Prove theorems that show "error → 0", as you increase computational effort.
- Use the analysis to develop fast methods.

Model problems
- Laplace equation: $-\Delta u = f$ in $\Omega \subseteq \mathbb{R}^n$, where $\Delta u = \partial_{x_1}^2 u + \partial_{x_2}^2 u + \dots + \partial_{x_n}^2 u$, plus some boundary conditions on $\partial\Omega$.
- Heat equation: $\partial_t u - \Delta u = f$ in $\Omega \subseteq \mathbb{R}^n$, $t > 0$.
  Steady state $(t \to \infty)$: $-\Delta u = f$.
- Wave equation: $\partial_t^2 u - \Delta u = f$ in $\Omega \subseteq \mathbb{R}^n$, $t > 0$.
- PLUS nonlinear variants e.g. $f = f(u, \nabla u)$. (Navier-Stokes)

Today : Laplace equation in dimension 1.

$$\left\{ \; -\partial_x^2 u = f \qquad \text{in} \qquad [0,1] \qquad (*) \right.$$

- Solution not unique : if u is a solution then

$$\{ u + c_1 x + c_2 \mid c_1, c_2 \in \mathbb{R} \}$$

is a 2-dim space of solutions. Add two boundary conditions :

$$\left\{ \; u(0) = 0 , \qquad u(1) = 0 \qquad (**) \right.$$

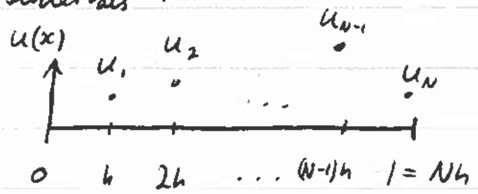
Then the boundary value problem $(*) + (**)$ has a unique solution.
How to compute it ?

ie. Given f, find u. How ?

Simplest approach : Finite differences = calculus.

$$\begin{cases} \partial_x u(x) = \lim\limits_{h \to 0} \dfrac{u(x+h) - u(x)}{h} \\[2ex] \partial_x^2 u(x) = \lim\limits_{h \to 0} \dfrac{u(x+h) + 2u(x) + u(x-h)}{h^2} \end{cases}$$

Fix $h > 0$, $h = \frac{1}{N}$, $N \in \mathbb{N}$, and divide interval $[0,1]$ into N subintervals :



Continuum | Discrete

$$\begin{cases} -\partial_x^2 u = f & \longleftrightarrow \quad \dfrac{-u^{n+1}+2u^n-u^{n-1}}{h^2} = f(x_n) \\ u(0)=0,\ u(1)=0 & \longleftrightarrow \quad u^0 = 0, \quad u^N = 0 \end{cases}$$

Linear System of equations: $A\vec{u} = \vec{F}$

$$F^n = f(x^n), \quad u^n \simeq u(x_n)$$

‖ (linear) PDE $\longrightarrow$ (linear) algebra.

Hope: $h \to 0$, $\lim_{h\to 0} u^n = u(x_n) \quad \forall n = 1,2,3,\dots, N = \frac{1}{h}$.

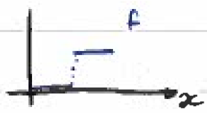
More concretely:

$$A\vec{u} = \frac{1}{h^2}\begin{pmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & 0 & \\ & -1 & 2 & -1 & & \\ & & -1 & 2 & -1 & \\ & 0 & & \ddots & \ddots & \\ & & & -1 & 2 & -1 \\ & & & & -1 & 2 \end{pmatrix}\begin{pmatrix} u^1 \\ u^2 \\ \vdots \\ \vdots \\ u^{N-1} \end{pmatrix}$$

Factor $\frac{1}{h^2} = N^2$

$$= \begin{pmatrix} f(x_1) \\ \vdots \\ f(x_{N-1}) \end{pmatrix} = \vec{F}$$

tri-diagonal matrix.

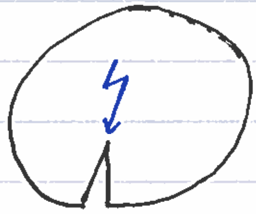
Q^n. - Does the discrete system $A\vec{u} = \vec{F}$ have a unique solution?
- How to solve $A\vec{u} = \vec{F}$ efficiently?
- What if f has a jump:  ? i.e. u not twice diff'ble.

$$\frac{u(x+h) - 2u(x) + u(x-h)}{h^2} \underset{h\to 0}{\not\longrightarrow} \text{"}\partial_x^2 u(x)\text{"}.$$

NOTE: Solutions in higher dimensions are often not smooth.
(eg. lightning striking a building / breaking waves).

Simple example: $\Delta u = 0$ in Ω =

Infinite electric field at tip.



Laplace equation revisited: same problem, weak formulation, energy, finite element method.

$$(*)\ \begin{cases} -\partial_x^2 u = f & \text{in } (0,1) \\ u(0) = u(1) = 0 \end{cases}$$

Thm. If u is sufficiently smooth then u is a solution to $(*)$ iff u minimises the "energy functional"

$$E(v) = \frac{1}{2}\int_0^1 (\partial_x v)^2\,dx - \int_0^1 f v\,dx$$

among all v with $\underbrace{\int v^2 < \infty, \int(\partial_x v)^2 < \infty}_{\text{"sufficiently smooth"}}$, $v(0) = v(1) = 0$ ▯

Remark: This theorem leads to a numerical method as follows.

Denote

$$H = \left\{ v : [0,1] \to \mathbb{R};\ \text{measurable},\ \int_0^1 v^2\,dx < \infty,\ \int_0^1 (\partial_x v)^2\,dx < \infty,\ v(0) = v(1) = 0 \right\}$$

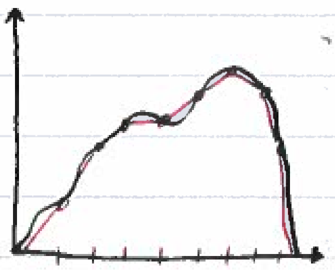
(CHECK: H is a vector space.)

Then the theorem says

$$\left\{ \text{Solve } (*) \iff \text{Find minimum of } E \text{ over } H \iff \text{minimise quadratic functional over } H. \right.$$

$h = \frac{1}{N}$, $N \in \mathbb{N}$

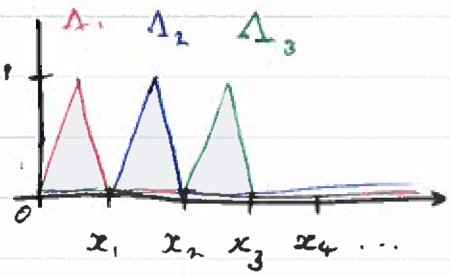
For $h > 0$, $H_h \subseteq H$ denotes the space of cts. functions in H which are linear over every subinterval (x_j, x_{j+1}), $x_j = jh$.



NOTE: H_h is a FINITE dim subspace of H. $\dim H_h = N-1$

Numerical method: minimise E over H_h. (finite dim minimisation prob.)

Key observation: every function $v_h \in H_h$ can be uniquely expressed as a sum of "hat functions", $\Lambda_i \in H_h$.



$$v_h(x) = \sum_{i=1}^{N-1} c_i \Lambda_i(x), \quad c_i = v_h(x_i).$$

Then

$$E(v_h) = \frac{1}{2}\int_0^1 \left(\sum_{i=1}^{N-1} c_i \partial_x \Lambda_i\right)^2 - \int_0^1 f \sum_{i=1}^{N-1} c_i \Lambda_i$$

$$= \sum_{i,j} \frac{1}{2} c_i c_j \underbrace{\boxed{\int_0^1 (\partial_x \Lambda_i)(\partial_x \Lambda_j)\, dx}}_{=: A_{ij}} - \sum_i c_i \underbrace{\boxed{\int_0^1 f \Lambda_i \, dx}}_{=: F_i}.$$

So we aim to minimise

$$\boxed{\frac{1}{2}\vec{c}^T A \vec{c} - \vec{F}\cdot\vec{c}}, \quad \vec{c} \in \mathbb{R}^{N-1}$$

then

$$\boxed{u_h = \sum c_j \Lambda_j}, \quad \text{is an approx. solution.}$$

NOTE: $$A = \frac{1}{h}\begin{pmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & 0 & \\ & -1 & 2 & -1 & & \\ & & & \ddots & & \\ & 0 & & -1 & 2 & -1 \\ & & & & -1 & 2 \end{pmatrix} = h \times \text{ matrix from finite difference.}$$

(see Finite Element Primer by D. Sylvester, or book by Braess.)

Moreover, $\int_0^1 f \Lambda_i \, dx \simeq h f(x_i)$.

Conclusion: Finite differences: $$\frac{1}{h^2}\begin{pmatrix} 2 & -1 & & \\ -1 & 2 & -1 & \\ & -1 & 2 & -1 \\ & & & \ddots \end{pmatrix} \vec{c} = \begin{pmatrix} f(x_1) \\ f(x_2) \\ \vdots \end{pmatrix}$$

Finite elements $\equiv$ minimisation of E over H_h,

$$\frac{1}{h}\begin{pmatrix} 2 & -1 & & \\ -1 & 2 & -1 & \\ & -1 & 2 & -1 \\ & & & \ddots \end{pmatrix} \vec{c} = \begin{pmatrix} \int f \Lambda_1 \, dx \\ \int f \Lambda_2 \, dx \\ \vdots \end{pmatrix}.$$

SAME PROBLEM upto small modification of RHS.

Finite elements give you a lot more freedom:

- can choose H_h differently:
  - piecewise polynomials of degree 2?
  - sines, cosines
  - x_j nonequidistant.