Higher-order polynomials on triangles

(p+1)(p+2)/2 points determine unique polynomial of degree p

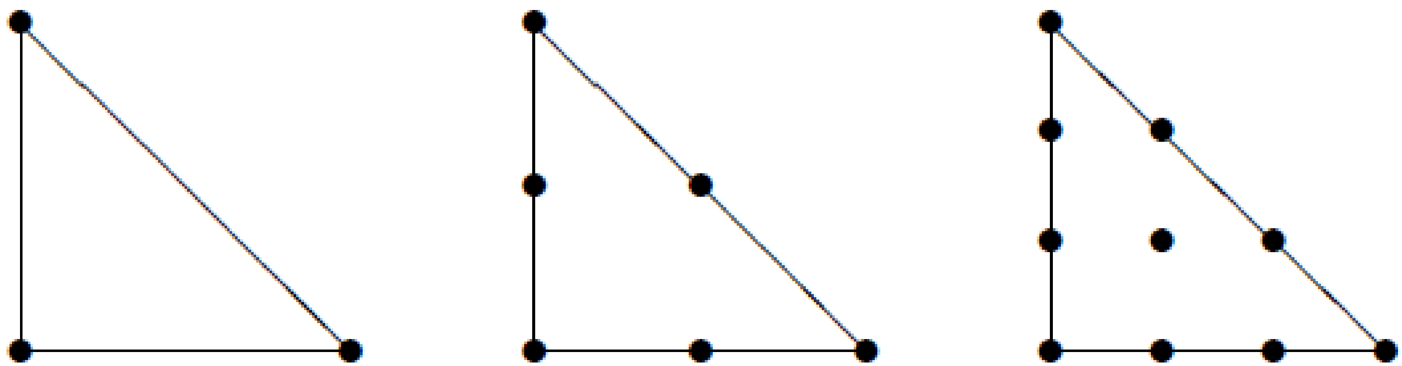


Fig. 16. Nodes of the nodal basis for linear, quadratic, and cubic triangular elements $\mathcal{M}_0^1$, $\mathcal{M}_0^2$, and $\mathcal{M}_0^3$

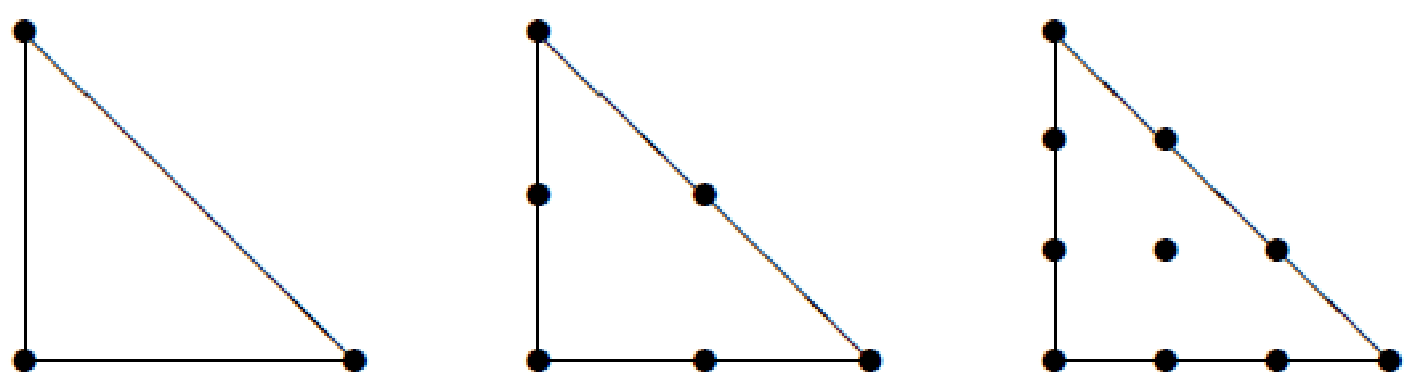
Higher-order polynomials on triangles

5.4 Remark. Let $t \geq 0$. Given a triangle T, suppose $z_1, z_2, \ldots, z_s$ are the $s = 1 + 2 + \cdots + (t+1)$ points in T which lie on $t+1$ lines, as in Fig. 16. Then for every $f \in C(T)$, there is a unique polynomial p of degree $\leq t$ satisfying the interpolation conditions

$$p(z_i) = f(z_i), \quad i = 1, 2, \ldots, s. \tag{5.3}$$

Proof. The result is trivial for $t = 0$. We now assume it has been established for $t-1$, and prove it for t. In view of the invariance under affine transformations, we can assume that one of the edges of T lies on the x-axis. Suppose it is the one containing the points $z_1, z_2, \ldots, z_{t+1}$. There exists a univariate polynomial $p_0 = p_0(x)$ with

$$p_0(z_i) = f(z_i), \quad i = 1, 2 \ldots, t+1.$$



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$$p_0(z_i) = f(z_i), \quad i = 1, 2 \ldots, t+1.$$

By the induction hypothesis, there also exists a polynomial $q = q(x, y)$ of degree $t - 1$ with

$$q(z_i) = \frac{1}{y_i}[f(z_i) - p_0(z_i)], \quad i = t+2, \ldots, s.$$

Clearly, $p(x, y) = p_0(x) + yq(x, y)$ satisfies (5.3). $\square$

Higher-order polynomials on triangles

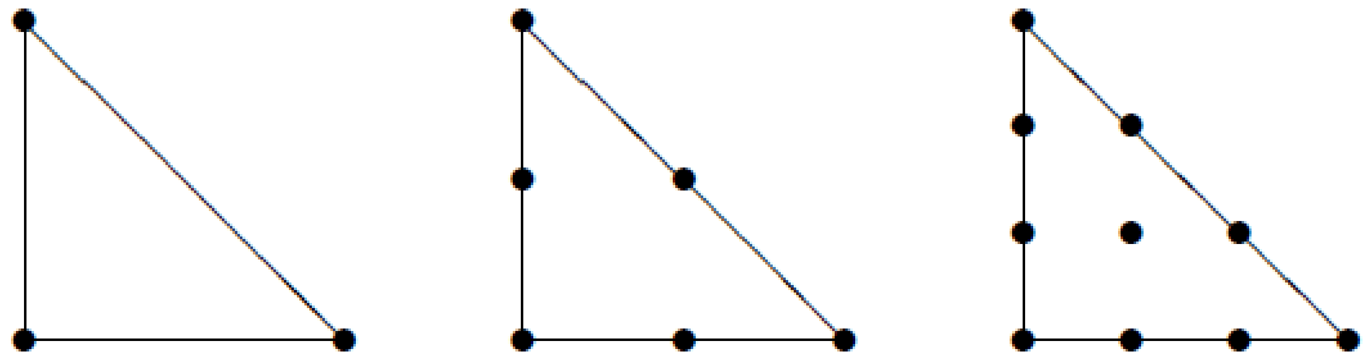


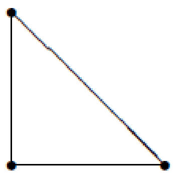
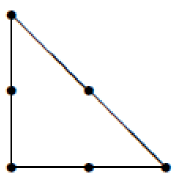
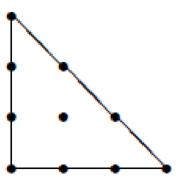
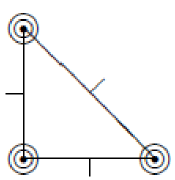
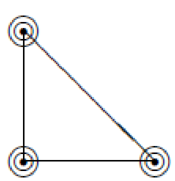
Fig. 16. Nodes of the nodal basis for linear, quadratic, and cubic triangular elements $\mathcal{M}_0^1$, $\mathcal{M}_0^2$, and $\mathcal{M}_0^3$

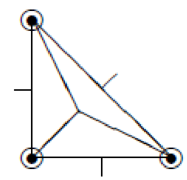
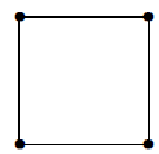
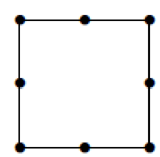
5.6 A Nodal Basis for C^0 Elements. Let $t \geq 1$, and suppose we are given a triangulation of Ω. In each triangle, we place $s := (t+1)(t+2)/2$ points as indicated in Fig. 16, so that there are $t+1$ points on each edge. By Remark 5.4, in each triangle a polynomial of degree $\leq t$ is determined by choosing values at these points. The restriction of any such polynomial to an edge is a polynomial of degree $\leq t$ in one variable. Now given an edge, the two polynomials on either side interpolate the same values at the $t+1$ points on that edge, and thus must reduce to the same one-dimensional polynomial. This ensures that our elements are globally continuous.

Other standard finite elements

Table 2. Interpolation with some standard finite elements

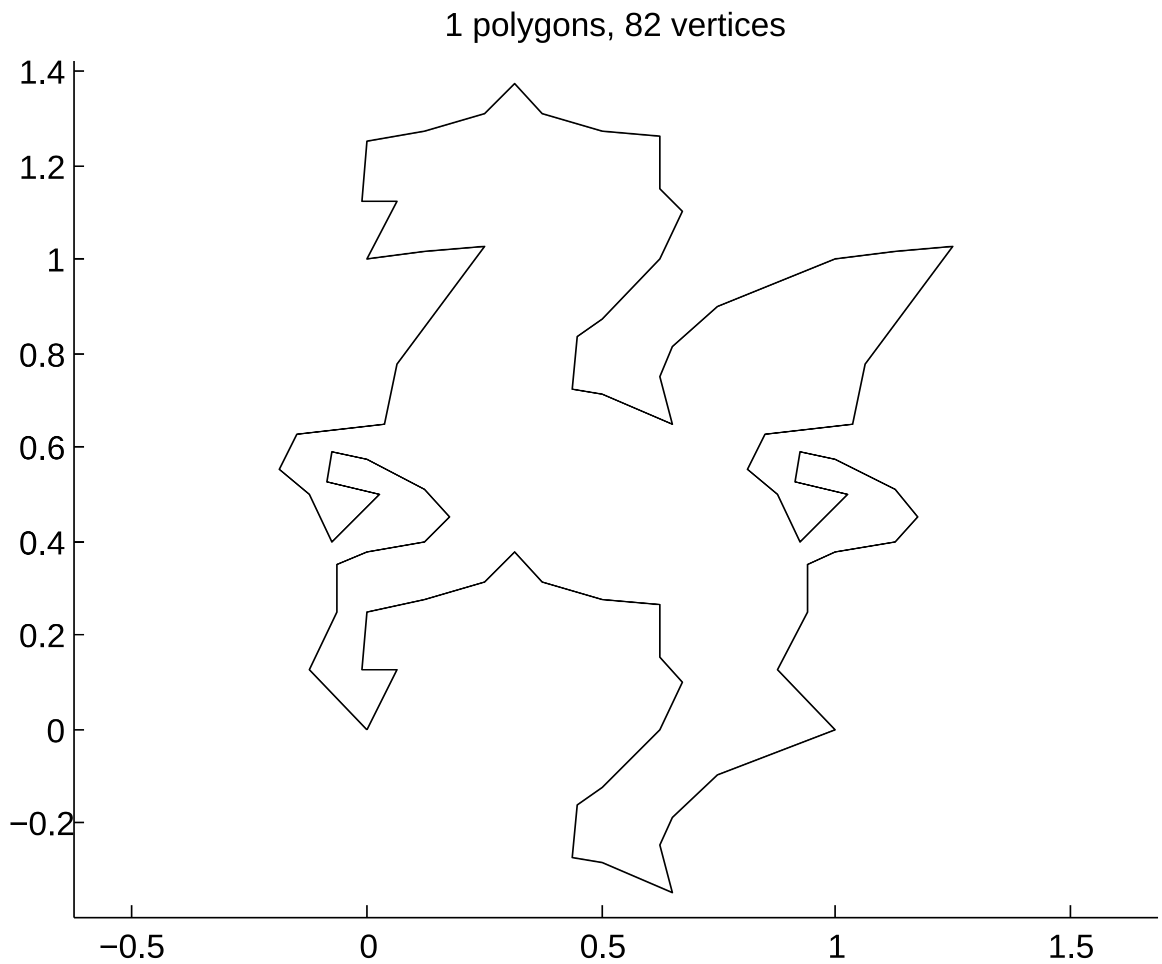
- • Function value prescribed
- ⊙ Function value and 1st derivative prescribed
- ◎ Function value and 1st and 2nd derivatives prescribed
- ⊥ Normal derivative prescribed

	Linear triangular element $\mathcal{M}_0^1$ $u \in C^0(\Omega)$ $\Pi_{\text{ref}} = \mathcal{P}_1, \quad \dim \Pi_{\text{ref}} = 3$
	Quadratic triangular element $\mathcal{M}_0^2$ $u \in C^0(\Omega)$ $\Pi_{\text{ref}} = \mathcal{P}_2, \quad \dim \Pi_{\text{ref}} = 6$
	Cubic triangular element $\mathcal{M}_0^3$ $u \in C^0(\Omega)$ $\Pi_{\text{ref}} = \mathcal{P}_3, \quad \dim \Pi_{\text{ref}} = 10$
	Argyris triangle $u \in C^1(\Omega)$ $\Pi_{\text{ref}} = \mathcal{P}_5, \quad \dim \Pi_{\text{ref}} = 21$
	Bell triangle $u \in C^1(\Omega)$ $\Pi_{\text{ref}} \subset \mathcal{P}_5, \ \partial_\nu u\|_{\partial T_i} \in \mathcal{P}_3, \quad \dim \Pi_{\text{ref}} = 18$

	Hsieh–Clough–Tocher element $u \in C^1(\Omega)$ $T = \bigcup_{i=1}^{3} K_i, \quad u\|_{K_i} \in \mathcal{P}_3, \quad \dim \Pi_{\text{ref}} = 12$
	Bilinear quadrilateral element Q_1 $u \in C^0(\Omega)$ $\Pi_{\text{ref}} \subset \mathcal{P}_2, \ u\|_{\partial T_i} \in \mathcal{P}_1, \quad \dim \Pi_{\text{ref}} = 4$
	Serendipity element $u \in C^0(\Omega)$ $\Pi_{\text{ref}} \subset \mathcal{P}_3, \ u\|_{\partial T_i} \in \mathcal{P}_2, \quad \dim \Pi_{\text{ref}} = 8$

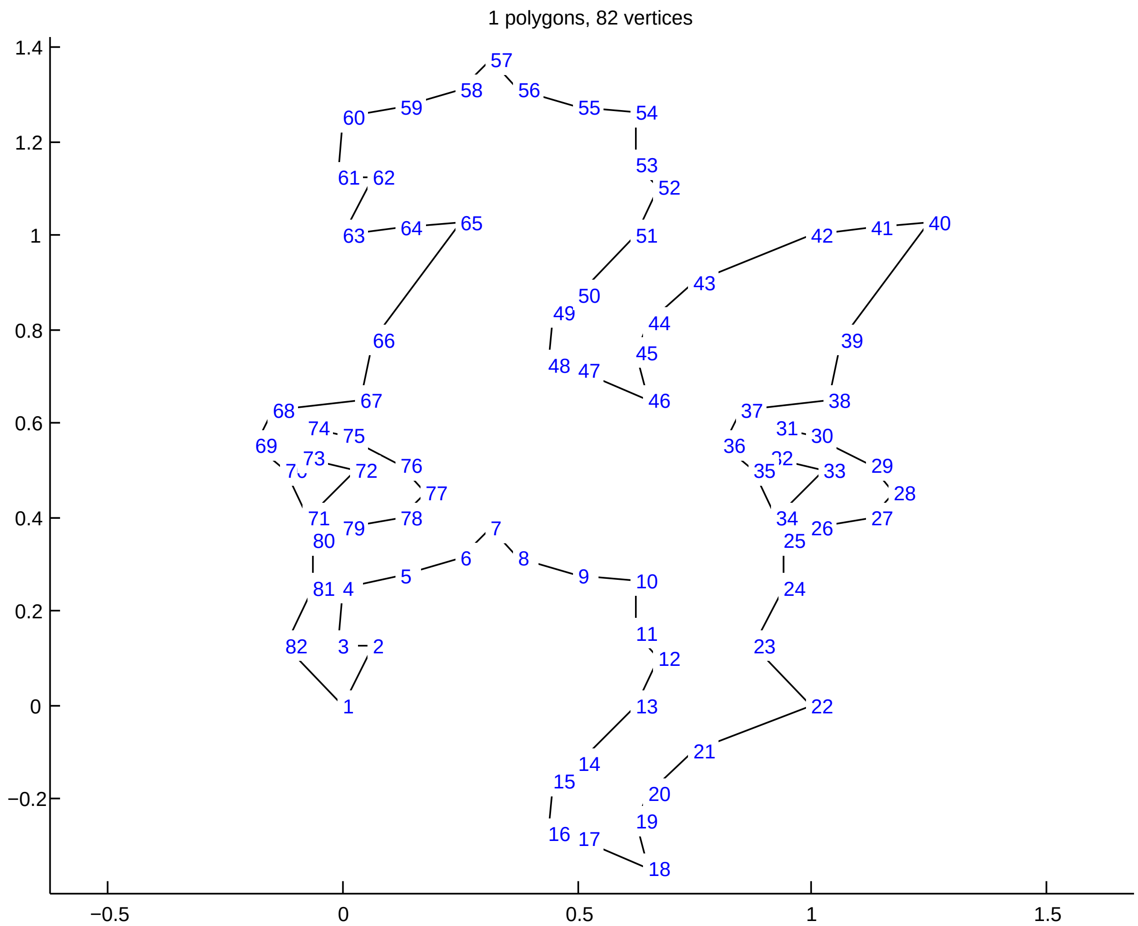
The next steps? (by M.C. Escher)

Going berserk (by A. Russo)



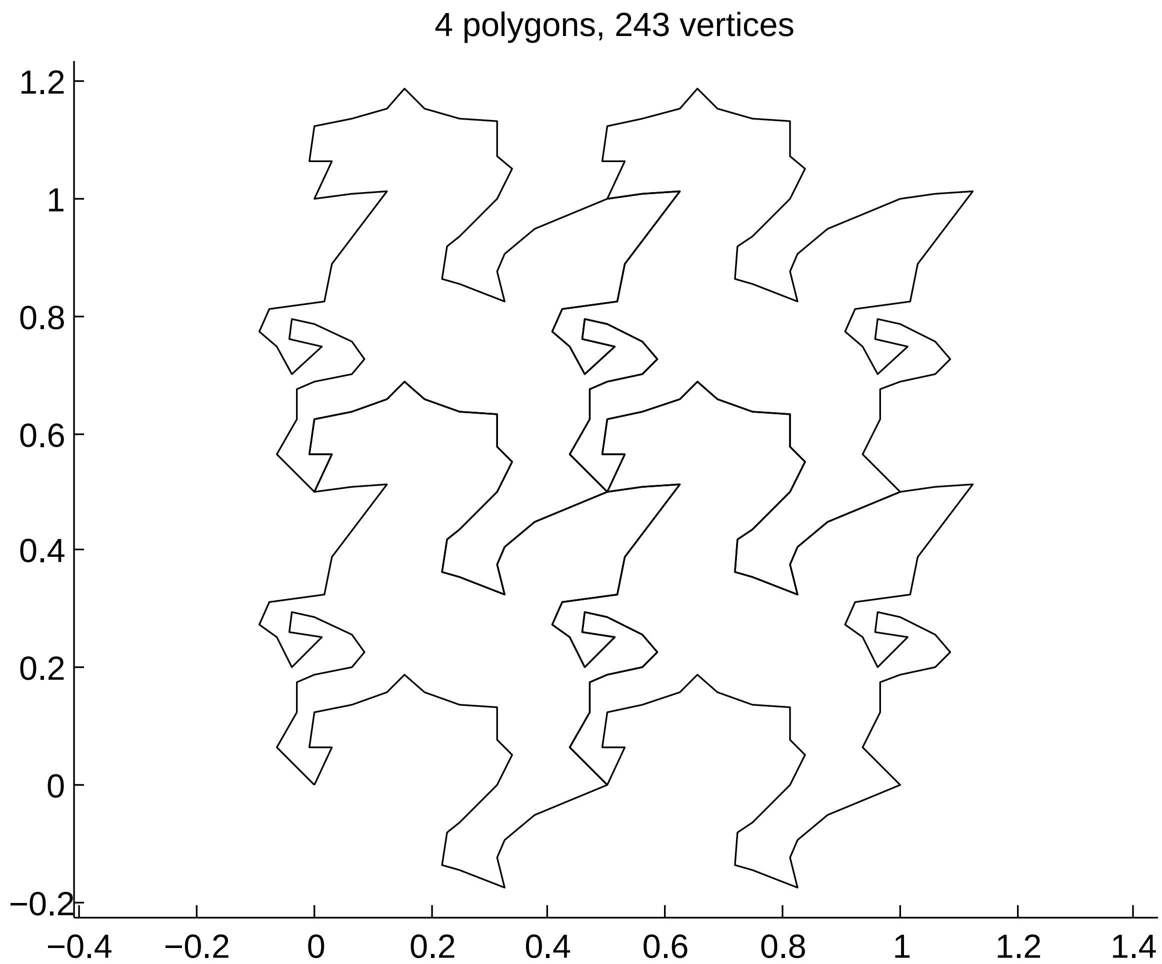
The first step: a pegasus-shaped polygon

Going berserk (by A. Russo)



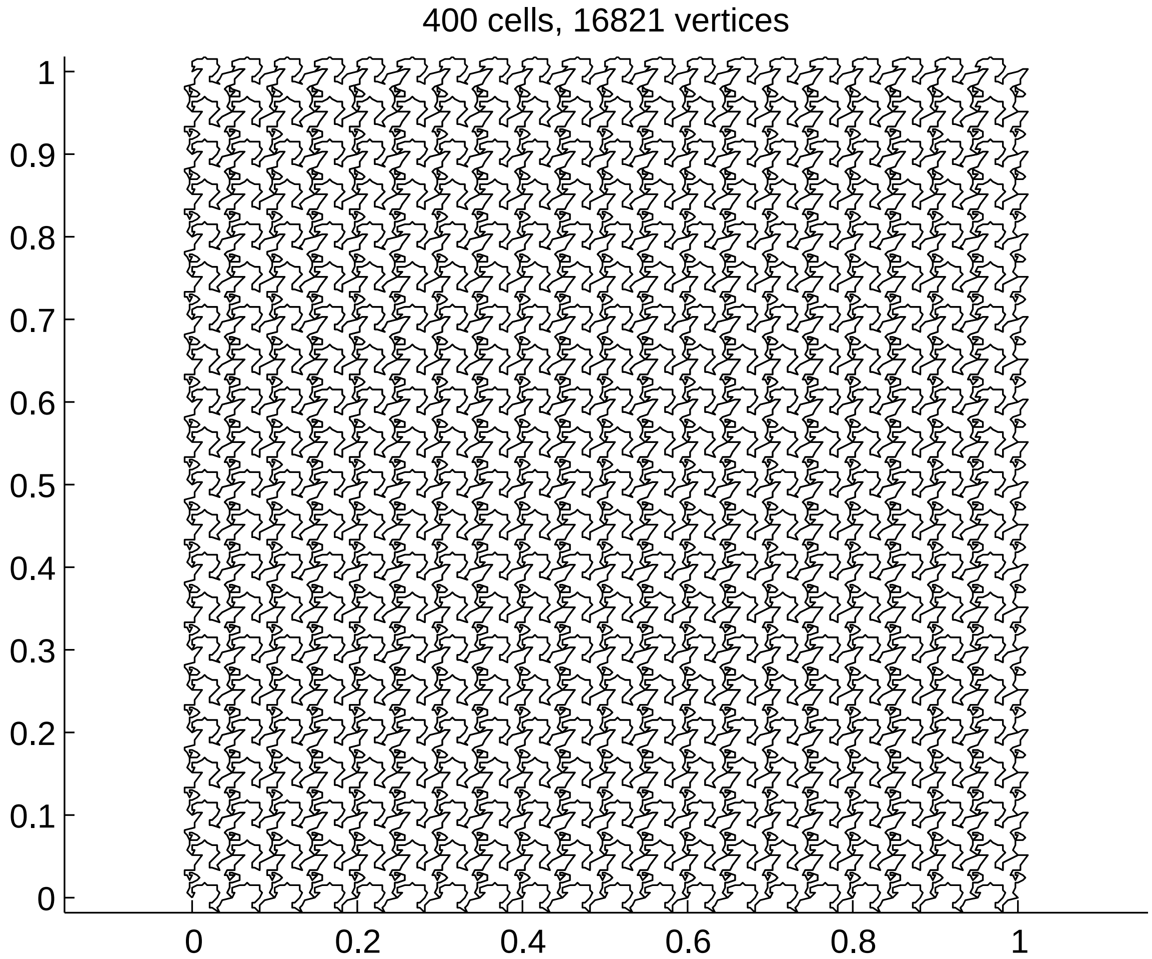
The second step: local numbering of nodes.

Going berserk (by A. Russo)



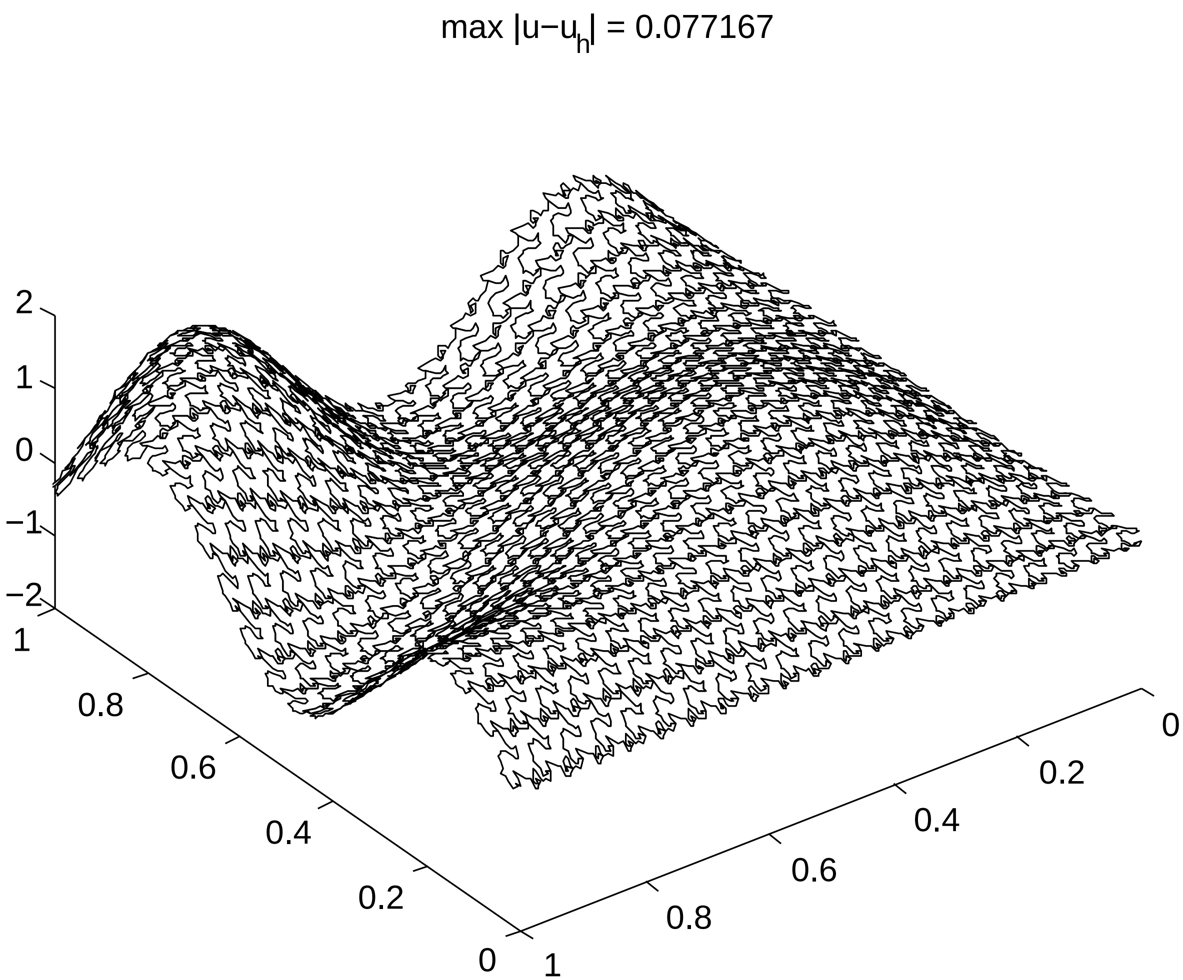
The third step: a mesh of 2×2 pegasus.

Going totally berserk (by A. Russo)



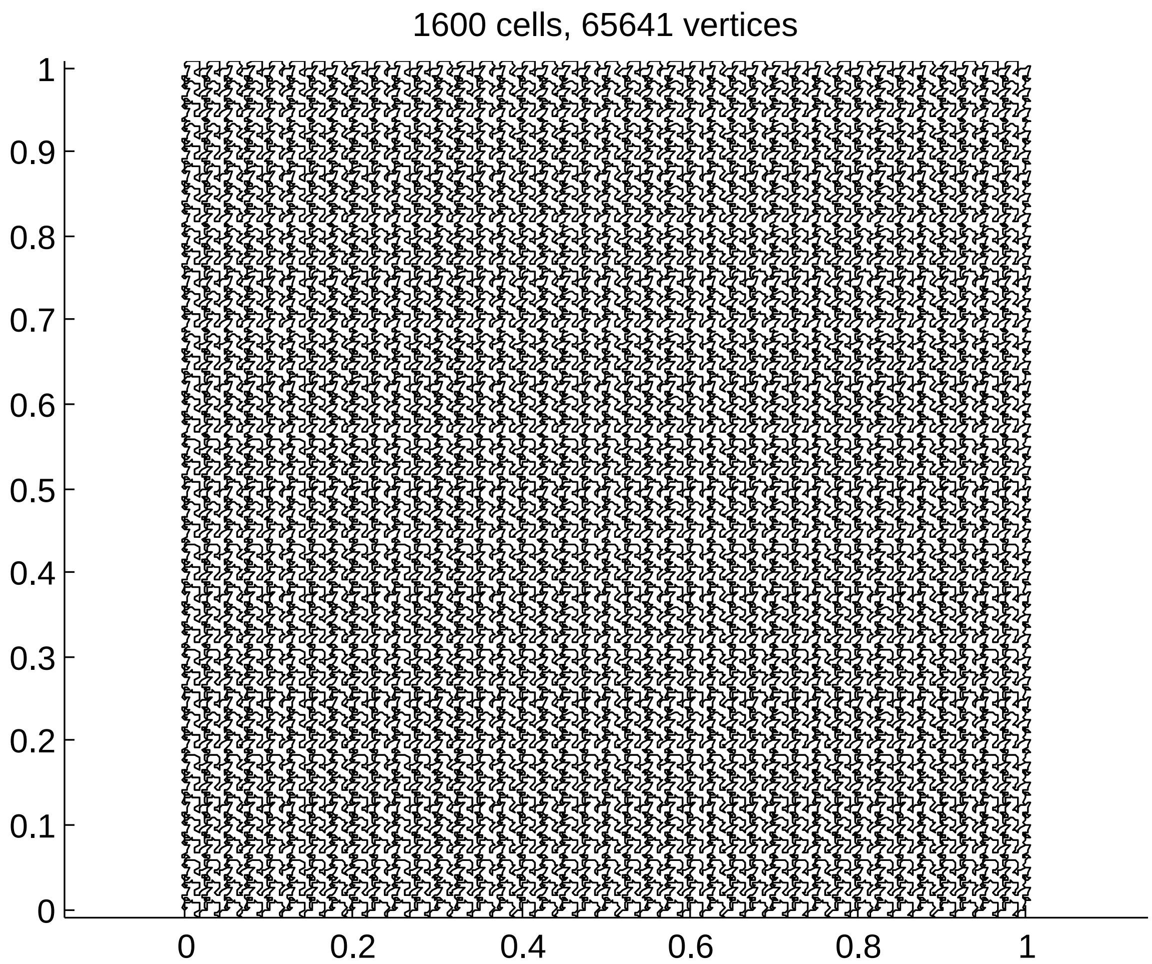
A mesh of 20×20 pegasus.

Going totally berserk (by A. Russo)



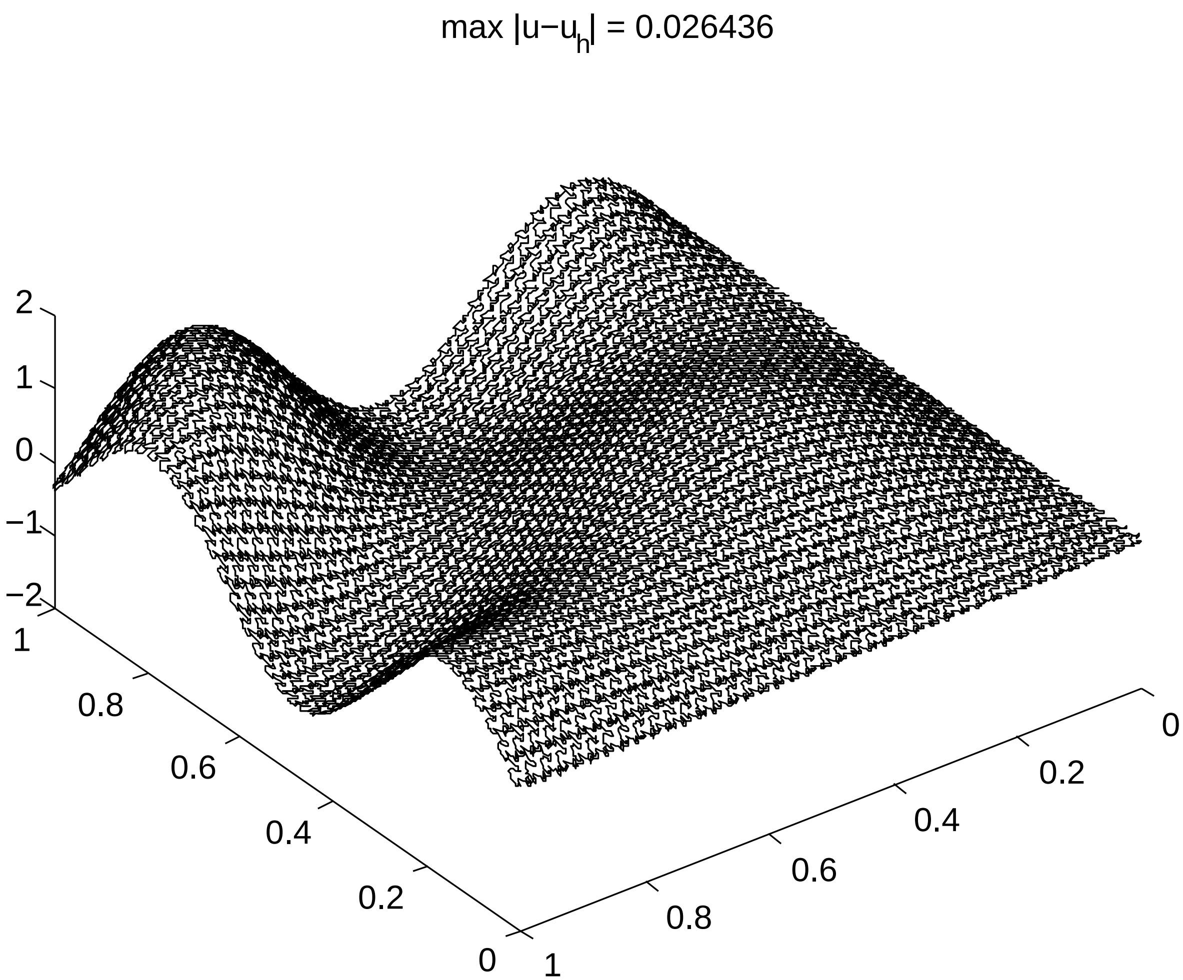
Solution on a 20 $\times$ 20-pegasus mesh.

Going **totally** berserk (by A. Russo)



A mesh of 40 $\times$ 40 pegasus.

Going **totally** berserk (by A. Russo)



Solution on a 40 $\times$ 40-pegasus mesh.