Finite difference methods — Analysis

Recall: $u : (0,1) \to \mathbb{R}$, $h > 0$ small

$$\partial_x u(x) \simeq \frac{u(x+h) - u(x)}{h}$$

$$\partial_x^2 u(x) \simeq \frac{u(x+h) - 2u(x) + u(x-h)}{h^2}$$

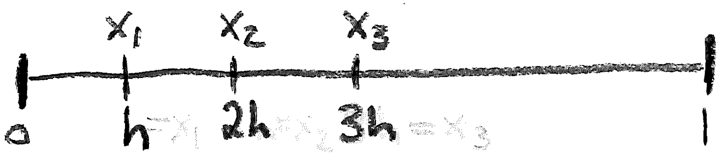
$$-\partial_x^2 u = f \quad \text{in } (0,1)$$

$$u(0) = u(1) = 0 \quad (\text{Dirichlet bc})$$

$$\Longrightarrow \quad \mathbb{A}\vec{c} = \vec{F}$$

where

$$\mathbb{A} = \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & -1 & \\ & & & \ddots & -1 \\ & & & -1 & 2 \end{bmatrix}$$



$$u(x_j) \simeq c_j$$

$$A\vec{c} = \vec{F} \quad , \quad c_j \simeq u(jh) \, , \; f_j = f(jh)$$

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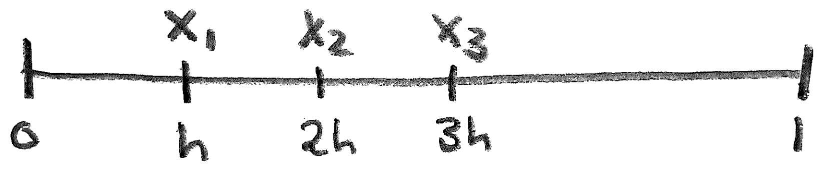

<u>Star Lemma:</u> $k \geq 1$. Suppose $(\alpha_i, p_i)_{i=0}^{k}$ are such that

- $\alpha_i < 0$ for $i = 1, \ldots, k$
- $\alpha_0 + \alpha_1 + \cdots + \alpha_k \geq 0$
- $\alpha_0 p_0 + \alpha_1 p_1 + \cdots + \alpha_k p_k \leq 0$
- Either $p_0 \geq 0$ or $\alpha_0 + \cdots + \alpha_k = 0$

Then if $p_0 = \max p_i \implies p_0 = p_1 = \cdots = p_k$.

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Apply to $A\vec{c} = \vec{F}$, with $F_i \leq 0$ for all i.

$$-c_0 + 2c_1 - c_2 = F_1, \dots, \qquad -c_{j-1} + 2c_j - c_{j+1} = F_j \leq 0 \quad (*)$$

If c_j is the maximum of the c's, let $p_0 = c_j$, $p_1 = c_{j-1}$, $p_2 = c_{j+1}$
$\alpha_0 = 2$, $\alpha_1 = -1$, $\alpha_2 = -1$

$$(*) \iff \alpha_0 p_0 + \alpha_1 p_1 + \alpha_2 p_2 = F_j \leq 0 \xRightarrow{\text{Star Lemma}} p_0 = p_1 = p_2. \text{ Repeat.}$$

also other assumptions fulfilled

<u>Conclusion:</u> If c_j is the maximum of the c's, then either

- j corresponds to a boundary node: $j = 0$ or N

or • c_j is constant: $c_0 = c_1 = \cdots = c_N$.

<u>Thm:</u> (<u>Discrete Maximum principle</u>)

a) Let $A\vec{c} = \vec{F} \leq 0$ be the finite difference discretisation of $-\partial_x^2 u = f$ in $(0,1)$. Then $\max_i c_i = \max\{u(0), u(1)\}$.
$u(0), u(1)$ given.

b) Assume the max is attained at an interior node.
Then $c_0 = c_1 = \cdots = c_N$.

See Braess, Thm. 3.5 in Chapter I for the Laplace equation in 2d.

Error equation:

- $-\partial_x^2 u(x) = f(x)$
- $-\partial_h^2 U_h(x_j) := \dfrac{U_h(x_{j+1}) - 2U_h(x_j) + U_h(x_{j-1})}{h^2} = f(x_j)$
- Assume $u(x_j) = U_h(x_j)$ for $x_j \in \partial\Omega$.

$$
\begin{aligned}
\Longrightarrow \quad -\partial_h^2 (u - U_h)(x_j) &= -\partial_h^2 u(x_j) - (A\vec{c})_j \\
(A\vec{c} = F) \qquad &= -\partial_h^2 u(x_j) - f(x_j) \\
(-\partial_x^2 u = f) \qquad &= -\partial_h^2 u(x_j) + \partial_x^2 u(x_j) =: -r_j^h
\end{aligned}
$$

$\Longrightarrow$ The error $\eta(x_j) = u(x_j) - U_h(x_j)$ satisfies

$$\boxed{\partial_h^2 \eta(x_j) = -r_j^h} \qquad \forall j.$$

Convergence: If $\underbrace{\boxed{r_j^h \xrightarrow{h \to 0^+} 0}}_{\text{consistency}}$ and $\underbrace{\boxed{(\partial_h^2)^{-1} \text{ bounded}}}_{\text{stability}}$ as $h \to 0^+$.

Theorem: (Convergence of Finite difference approximations)

Suppose the exact solution u is twice continuously differentiable and that $\partial_x^2 u$ is uniformly continuous in $(0,1)$.

Then
$$\max_{x_j} |U_h(x_j) - u(x_j)| \longrightarrow 0 \quad \text{as } h \to 0^+.$$

The proof requires:

Lemma: Let V_h be the solution to $-\partial_h^2 V_h(x_j) = 1 \quad \forall j$

$$V_h(0) = V_h(1) = 0$$

Then $0 \leq V(x_j) \leq \frac{1}{2} x(1-x) =: w(x)$

Proof: Set $W_j = w(x_j)$. Then $-\partial_h^2 W = -\partial_x^2 W = 1$, because error of $|\partial_h^2 W - \partial_x^2 W| \leq Ch^3$ (Taylor expansion).
exact!

Also $W \geq 0$. Discrete max. principle $\Longrightarrow$ $0 \leq V \leq W$ $\square$

Proof of Thm: (same proof works in 2d, see Braess I.4)

- Taylor expansion at point x_j: $-\partial_h^2 u(x_j) = -\partial_x^2 u(\xi)$ for some ξ

- Uniform continuity of $\partial_x^2 u$: $|-r_j^h| = |-\partial_h^2 u(x_j) + \partial_x^2 u(x_j)|$

$$= |-\partial_x^2 u(\xi) - \partial_x^2 u(x_j)|$$

$$\Rightarrow \max_j |r_j^h| \xrightarrow[h \to 0]{} 0$$

- Error equation: $\partial_h^2 \eta(x_j) = -r_j^h$, where $\eta(x_j) = u(x_j) - U_h(x_j)$

$$\leq \max_j |r_j^h| = \text{constant}$$

Lemma

$$\Longrightarrow \max_j |\eta(x_j)| \leq \underbrace{\max_{x \in (0,1)} \left[\tfrac{1}{2} x(1-x)\right]}_{= \frac{1}{8}} \cdot \max_j |r_j^h| \longrightarrow 0$$

$\boxed{}$