Plan for today: • Stability of time stepping 06.03.18

• Variational methods for time dep. PDEs ①

• Saddle point problems

Last time: Time stepping Imp./Exp. Euler → Error: dt

Crank-Nicolson → Error: $(dt)^2$

Thm: Stability: if $|y_0^{num} - y_0| < \delta$ and num. error in each step $< \delta$ then Error $\leq C\delta$ for all times.

Convergence rate: If Error of approx $y \simeq (dt)^p$

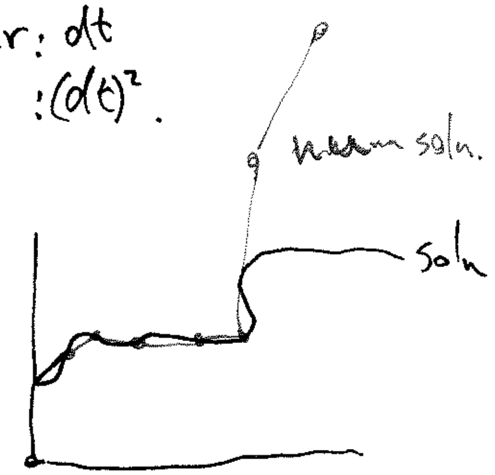
Then $|y(t_i) - y^{num}(t_i)| \leq C\,(dt)^p$.

• Error of explicit euler: dt

implicit euler: dt

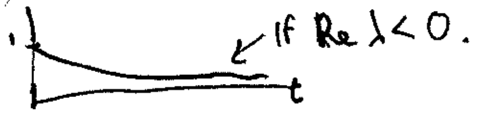
C-N : $(dt)^2$.

Typical problem:



Absolute stability: Take testproblem $y'(t) = \lambda y(t)$, $\lambda \in \mathbb{C}$.
$y(0) = 1$

Exact soln.: $y(t) = e^{\lambda t}$.



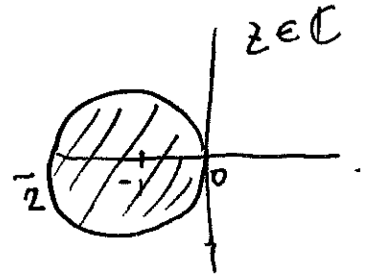
Num. method is absolutely stable if $(y_n) \to 0$ as $n \to \infty$.

This depends on λ, dt. So, Region of abs. stability

$$A = \{ z \in \mathbb{C} : z = \lambda dt : |y_n| \to 0 \text{ as } n \to \infty \}$$

Explicit Euler: $y_n = y_{n-1}(1 + \lambda dt)$
$y_0 = 1$

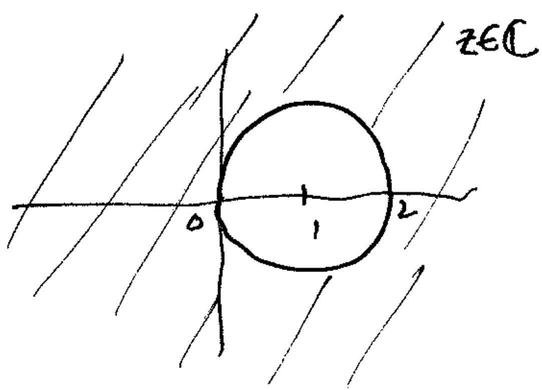
$\rightsquigarrow \quad y_n = (1 + \lambda dt)^n y_0 = (1 + \lambda dt)^n$



This is stable provided $|1 + \lambda dt| < 1$.

Implicit Euler: $y_n = \left(\frac{1}{1 - \lambda dt} \right)^n y_0$

Stable provided $\frac{1}{|1 - \lambda dt|} < 1 \rightsquigarrow |1 - \lambda dt| > 1$



Upshot: RAS for Imp. Euler contains Left half plane, where exact Soln. $\to 0$.

In practice Explicit methods require very small time steps.

Thm: There are no explicit unconditionally absolutely stable schemes.

All these methods work for time dependent PDEs.

Ex.: $\begin{cases} \partial_t u - \Delta u = f & \text{in } \Omega \\ u(t=0) = u_0 & \text{in } \Omega \end{cases}$

$$\rightsquigarrow \simeq \frac{u(t+dt) - u(t)}{dt} - \Delta u(t) = f(t).$$

$$\rightsquigarrow u(t+dt) = u(t) + dt\, \Delta u(t) + dt \cdot f$$

Discretize in space: Weak formulation

$$\int_\Omega v(x)\, u(t+dt, x)\, dx = \int_\Omega u v - dt \int_\Omega \nabla u \nabla v + dt \int_\Omega f v \quad \leftarrow (x,t)$$

Choose $H_h \subset H_0^1(\Omega)$ with basis $\{\phi_1 \dots \phi_h \dots\}$

$$u_h = \sum c_j \phi_j \quad , \quad v = \phi_k$$

$$\sum_j c_j \int \phi_k(x) \phi_j(x)\, dx = \sum_j c_j \int_\Omega \phi_k \phi_j - \sum_j c_j\, dt \int_\Omega \nabla\phi_k \nabla\phi_j + F\text{-term}.$$

$(t+dt)$

$$\Rightarrow \quad M \vec{c}(t+dt) = M \vec{c}(t) - dt\, A \vec{c}(t) + F\text{-term}$$

$M_{ij} = \int_\Omega \phi_j(x) \phi_i(x)\, dx$ — mass matrix

$A_{ij} = \int_\Omega \nabla\phi_i(x) \nabla\phi_j(x)\, dx$ — stiffness matrix

Any method for ODEs can be translated to heat eq. 4.

Space-time variational methods

$$\partial_t u - \Delta u = f(t) \quad , \quad u(t=0) = u_0 .$$

Multiply by $v(t,x)$. Integrate ove space and time.

$$\int_0^T \int_\Omega \partial_t u_h v_h \, dx\,dt + \int_0^T \int_\Omega \nabla u_h \cdot \nabla v_h \, dx\,dt = \int_0^T \int_\Omega f(x,t)\, v_h \, dx\,dt$$

$\forall v_h \in Y_h$. Find $u_h \in X_h$.

$$X_h = \left\{ y_h \in L^2(0,T; H_0^1(\Omega)) : y_h|_{[t_m, t_{m+1}]} \text{ is polynomial of degree } 1 \text{ in time and } 1 \text{ in space}, \; y_h \text{ continuous}, \; y_h(0) = u_0 . \right\}$$

$$Y_h = \left\{ y_h \in L^2(0,T, H_0^1(\Omega)) : y_h|_{[t_m, t_{m+1}]} \text{ as below}, \; y_h \text{ cont in space} . \right\}$$

Result: Matrix equation

$$A\vec{c} = F \quad \text{in space time}$$

Problem:
- Need to solve for all times in one step.
- Needs lots of memory. No time stepping.
- For nonlinear problems, need to know Time of
- existence of solutions.

Improvement: Discontinuous Galerkin in time.

In special cases, this recovers implicit Euler method and higher-order methods.

⊕ Lead to time-stepping schemes

⊕ very flexible for refinements.

Idea: Use Y_h for both test and trial functions.

Problem: How to define $\partial_t u_h$?

Clear for $\partial_t u_h$ in (t_m, t_{m+1}).

At the kinks? We don't care and add appropriate counter terms to weak form.

DG-weak form:

$$\sum_{m=1}^{M} \int_{t_m}^{t_{m+1}} \int_{\Omega} \partial_t u_h \, v_h \, dx\,dt + \int_{t_m}^{t_{m+1}} \int_{\Omega} \nabla u_h \cdot \nabla v_h \, dx\,dt + \sum_{m=1}^{M} \int_{\Omega} [u_h]^m_\tau v_+^m \, dx$$

$$= \sum_{m=1}^{M} \int_{t_m}^{t_{m+1}} \int_{\Omega} f \, v_h \, dx\,dt$$

For $u_h \in \mathbb{P}_0(I_m, H_h)$ recovers implicit Euler.