Next week: Lister L+T Centre (8)

Past weeks: $-\Delta u = f$ in Ω.
and more general

This week: Time dependent problems

$$\partial_t u = \Delta u + f(u) = F(u) \quad (*) \qquad \partial_t y = f(t,y)$$

Basic fact: • $(*)$ admits a solution provided f is continuous (Peano)

• Solution is unique provided in addition $y \mapsto f(t,y)$ is Lipschitz $(\forall t)$ (Picard-Lindelöf)

Idea of proof (Peano):

$$\dot{y}(0) = \frac{y(dt) - y(0)}{dt} = f(0, y(0)) \quad \Rightarrow \text{solve for } y(dt): \quad y(dt) \simeq y(0) + dt \cdot f(0, y(0))$$

Repeat: $y(k\,dt) \simeq y((k-1)dt) + dt\, f((k-1)dt, y((k-1)dt))$

Show: this converges for $dt \to 0^+$

Explicit Euler method

Basic problems: • Convergence
• Stability

3 similar methods:

forward difference quotient

$$\dot{y}(t) \simeq \frac{y(t+dt) - y(t)}{dt}$$

• Explicit Euler: $y^{k+1} = y^k + dt\, f(kdt, y^k)$

$$\dot{y}(t) \simeq \frac{y(t) - y(t-dt)}{dt}$$

• Implicit Euler: $\dfrac{y^k - y^{k-1}}{dt} \simeq \dot{y}(kdt) = f(kdt, y^k)$

need to solve equation for y^k

$$\Rightarrow \underbrace{y^k - dt\, f(kdt, y^k) = y^{k-1}}_{\text{This will not blow-up}}$$

• Average of explicit + implicit
(central difference quotient)

$$\dot y(t) \approx \frac{y(t+dt) - y(t-dt)}{2dt}$$

$\Downarrow$

Crank-Nicolson method: $\frac{y^{k+1} - y^{k}}{dt} = \text{"} f((k+\tfrac{1}{2})dt, y^{k+1/2})\text{"}$

more accurate
faster convergence

$$\approx \frac{1}{2}\left(f(kdt, y^k) + f((k+1)dt, y^{k+1})\right)$$

All of the above are 1-step methods
y^{k+1} only depends on y^k, but not on earlier y's.

m-step method: $y^k, y^{k-1}, \dots, y^{k-m+1}$

Convergence: A method is of order p if the exact solution $y(t)$ satisfies the defining equations up to an error of size $(dt)^{p+1}$.

More formally the previous methods (or any 1-step method) are of the form

$$y^{k+1} + \phi(y^{k+1}, y^k, f, t_k, dt) = 0$$

order p: $\left|y((k+1)dt) + \phi(y((k+1)dt), y(kdt), f, kdt, dt)\right| \le C(dt)^{p+1}$

A method is consistent if this actually goes to 0 faster than dt

Order p, $p>0 \Rightarrow$ consistent

Examples:

• Explicit Euler: $\left|\underbrace{y((k+1)dt) - y(kdt)}_{= \dot y(kdt)dt + \frac{1}{2}\ddot y(kdt)dt^2 + \dots} - dt f(kdt, y(kdt))\right|$

y satisfies $\dot y = f(t,y)$ $\;= \cancel{f(kdt, y(kdt))dt} + \frac{1}{2}\ddot y(kdt)$

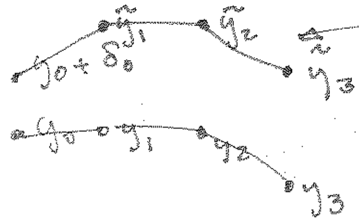
$\le C(dt)^2$: Order = 1

2

· Implicit Euler : Order 1.

· Crank-Nicolson : Order 2

<u>Zero-stability</u> : "stability around y^0" phrased for the discretised method



$\tilde{y}_k$ satisfy recurrence with error δ_k.

If $y_0 \simeq \tilde{y}_0 = y_0 + \delta_0$

$\overset{?}{\Rightarrow} y_k \simeq \tilde{y}_k$?

$\tilde{y}$: initial condition + recurrence have error of size $\leq \varepsilon$

y : exact initial cond + exact recurrence

<u>Zero stability</u> : $\forall dt$ small $|\tilde{y} - y| \leq C\varepsilon$

<u>Thm</u> : • $\dot{y} = f(t, y)$, $y(0) = y_0$

• Explicit method $y_{k+1} = y_k + dt\,\phi(kdt, y_k, f(kdt), dt)$

• Assume ϕ is Lipschitz continuous w.r.t. y_k

$$|\phi(\cdot, y_k, \cdot, \cdot) - \phi(\cdot, \tilde{y}_k, \cdot, \cdot)| \leq \Lambda\, |y_k - \tilde{y}_k|$$

$\rightarrow$ Method is <u>zero-stable</u> :

<u>Proof</u> : Start from eqns satisfied by $\tilde{y}$ and y

$$\underbrace{\tilde{y}_{n+1} - y_{n+1}}_{= w_{n+1}} = \underbrace{\tilde{y}_n - y_n}_{= w_n} + dt\left[\phi(\dots \tilde{y}_n \dots) + \delta_{n+1} - \phi(\dots y \dots)\right]$$

$$|\phi(\dots \tilde{y}_n \dots) - \phi(\dots y \dots)| \leq \Lambda\, |\tilde{y}_n - y_n| = \Lambda\, |w_n|$$

$$|w_{n+1}| \leq |w_n| + \Lambda\, |w_n|\, dt + dt\, \delta_{n+1}$$

<u>Iterate</u> : $|w_{n+1}| \leq |w_{n-1}| + dt\, \Lambda\, |w_{n-1}| + dt\, \delta_n$

$+ dt\, \Lambda\, |w_{n-1}| + dt^2 \Lambda^2 |w_{n-1}|$

$+ dt^2 \Lambda\, \delta_n + dt\, \delta_{n+1}$

$$w_{n+1} = w_n + dt\left(\phi(..\tilde{y}_n...) - \phi(..y_n..) + \delta_{n+1}\right)$$

$$\overset{!}{=} w_{n-1} + dt\left(\phi(..\tilde{y}_{n-1}..) - \phi(...y_{n-1}..) + \delta_n\right)$$

$$\ldots$$

$$= w_0 + \sum_{j=0}^{n} dt\, \delta_{j+1} + dt \sum_j \left(\phi(..\tilde{y}_j...) - \phi(..y_j..)\right)$$

$$\boxed{|w_{n+1}| \leq |w_0| + \sum_{j=0}^{n} dt\, |\delta_{j+1}| + \Lambda\, dt \sum_j |w_j|}$$

Need upper bound for w_{n+1} from $[\ldots]$

Answer: Gronwall's lemma (in a discrete version)

$$[\ldots] \Rightarrow |w_{n+1}| \leq (1 + dt(n+1))\, e^{(n+1)dt\Lambda}\, \varepsilon, \quad \varepsilon = \max|\delta_j|$$

Thm: Under the same assumptions as in the previous theorem the error is

$$|y(ndt) - y^n| \leq \left(|y(0) - y^0| + \underbrace{n(dt)^{p+1}}_{(ndt)(dt)^p}\right) e^{ndt\Lambda}$$

$y(ndt)$: exact sol. at time ndt; y^n: numerical sol.

where p is the order of the method

Proof: $y_{k+1} = y_k + dt\,\phi(kdt, y_k, f(kdt), dt)$

exact sol. of order p:

$$y((k+1)dt) = y(kdt) + dt\,\phi(kdt, y(kdt), f, dt)$$

$$w_{k+1} = y_{k+1} - y((k+1)dt) = \underbrace{y_k - y(kdt)}_{w_k} + dt\left(\phi(..y_k..) - \phi(...y(dtk)..)\right)$$

$$w_{k+1} = w_0 + dt \underbrace{\sum_j \left(\phi(y_j) - \phi(y(j\,dt))\right)}_{\leq \Lambda |y_j - y(jdt)| = \Lambda |w_j|}$$

Discrete Gronwall $\Rightarrow$ assertion $\blacksquare$