Lecture 5 Numerical Analysis 13/02

Basic Result : Cea's Lemma

- $a : H \times H \longrightarrow \mathbb{R}$ bilinear form.
- $a(u,v) \leq C\|u\|\|v\|$ continuity
- $a(u,u) \geq k\|u\|^2$ coercivity.

Choose $H_h \subseteq H$, F.E: $a(u_h, v_h) = f(v_h)$, $\forall v_h \in H_h$

P.D.E: $a(u,v) = f(v)$, $\forall v \in H$.

$$\|u - u_h\| \leq \frac{C}{k} \inf \|u - v_h\|, \quad v_h \in H_h$$

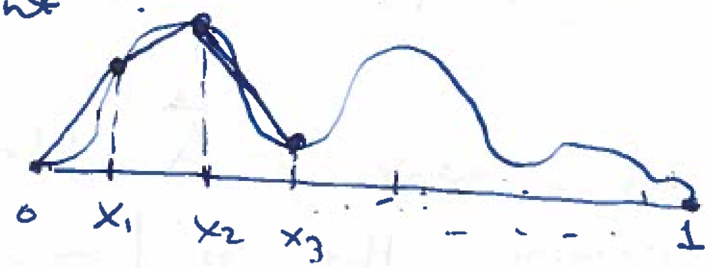
$H_h = \{$ p.w. linear functions, associated to a mesh $\}$

$= \{$ p.w. polynomials of degree p associated to a mesh

$= \{ \sin, \cos$, etc. . . $\}$.

P.w. linear functions : determined by their values in the nodes x_j.

A polynomial of degree 2 is uniquely determined by the values in the end points + $(p-1)$ interior (auxilary) point



dim H_h = # auxilary nodes + # of the x_j's.

Solve $A\vec{c} = \vec{F}$ $(A = A^T, A > 0)$

Consider $E(\vec{c}) = \frac{1}{2}\vec{c}^{\,T} A\vec{c} - \vec{F}\cdot\vec{c} = \frac{1}{2}\vec{c}\cdot(A\vec{c}) - \vec{F}\cdot\vec{c}$

By the gradient test $\frac{\partial E}{\partial c_j} = 0$ for all j.

$A\vec{c} = \vec{F}$

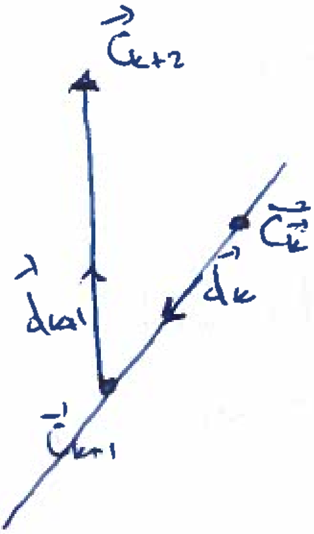
The Equation we are trying to solve $\Longleftrightarrow$ Finding a minimizer

Idea: Flow along $\vec{\nabla} E(\vec{c})$

Algorithm: Choose $\vec{c}_0 \in \mathbb{R}^d$ (initial guess)

- Determine direction $\vec{d}_k = -\nabla_{\vec{c}} E(\vec{c}_k)$
- Line search: Determine the minimum of $E(\vec{c})$

Some explicit formulas: $E = \frac{1}{2}\vec{c}^{\,T} A\vec{c} - \vec{F}\cdot\vec{c}$



$$\vec{d}_k = \vec{F} - A\vec{c}_k$$

$$\vec{c}_{k+1} = \vec{c}_k + \left(\frac{\vec{d}_k\cdot\vec{d}_k}{\vec{d}_k\cdot(A\vec{d}_k)}\right)\vec{d}_k$$

$$\vec{d}_k = -\nabla_{\vec{c}_k} E(\vec{c}_k)$$

$\vec{c}_{k+1}$:= the minimizer in this direction $\vec{d}_k$.

Condition number of A: $k(A) = \frac{\lambda_{max}}{\lambda_{min}}$

and determines the speed of convergence.

example: $A = \begin{pmatrix} 0.1 & 0 \\ 0 & 100 \end{pmatrix} \Rightarrow k(A) = 1000$.

As a measure of the error, we use energy norm

$$\|x\|_A := (\underbrace{\vec{x}^T A \vec{x}}_{>0})^{1/2} > 0, \text{ if } \vec{x} \neq 0 \text{ (because } A > 0)$$

If $\vec{c}^*$ is the exact solution $(A\vec{c}^* = \vec{f} \Leftrightarrow E(\vec{c}) \geq E(\vec{c}^*))$

$$\Rightarrow E(\vec{c}) = E(\vec{c}^*) + \frac{1}{2}\|\vec{c} - \vec{c}^*\|_A^2 .$$

$$E(\vec{c}_{k+1}) = E\left(\vec{c}_k + \frac{\vec{d}_k \cdot \vec{d}_k}{\vec{d}_k \cdot A\vec{d}_k} \vec{d}_k\right) = E(\vec{c}_k + a_k \vec{d}_k)$$

$$= \frac{1}{2}(\vec{c}_k + a_k \vec{d}_k)^T A(\vec{c}_k + a_k \vec{d}_k) - \vec{F} \cdot (\vec{c}_k + a_k \vec{d}_k)$$

$$= E(\vec{c}_k) + a_k \vec{d}_k^T (A\vec{c}_k - \vec{F}) + \frac{1}{2} a_k^2 \vec{d}_k A \vec{d}_k$$

$$= E(\vec{c}_k) - \frac{1}{2}\frac{(\vec{d}_k \cdot \vec{d}_k)^2}{\vec{d}_k \cdot (A\vec{d}_k)} \Longleftarrow *$$

< 0

Kantovich Inequality

$$\frac{(x^T A x)(x^T A^{-1} x)}{(x^T x)^2} \leq \left(\frac{1}{2}\sqrt{\kappa_A} + \frac{1}{2}\sqrt{\kappa_A}\right.$$

$$\vec{d}_k := \vec{F} - A\vec{c}_k = -A(\vec{c}_k - \vec{c}^*)$$

$$\|\vec{c}_k - \vec{c}^*\|_A^2 = (\vec{c}_k - c^*)^T A(\vec{c}_k - \vec{c}^*) \qquad = -A^{-1}\vec{d}_k$$

$$= (A^{-1}\vec{d}_k)^T A (A^{-1}\vec{d}_k)$$

$$= \vec{d}_k^T A^{-T}(AA^{-1})\vec{d}_k$$

$$= \vec{d}_k^T A^{-1} \vec{d}_k$$

$$\Rightarrow \|\vec{c}_{k+1} - \vec{c}_k^*\|_A^2 = \|\vec{c}_k - \vec{c}^*\|_A^2 \left[1 - \frac{(\vec{d}_k \cdot \vec{d}_k)^2}{(\vec{d}_k^T A \vec{d}_k)(\vec{d}_k^T A^{-1} \ldots)}\right]$$

Error after k-steps.

Use $*$ $\vec{x} = \vec{d}_k$: $\|\vec{c}_{k+1} - \vec{c}^*\|_A \leq \|\vec{c}_k - \vec{c}_*\|_A \left(1 - \frac{4}{(\kappa^{1/2} + \kappa^{\ldots}}\right.$

Theorem: The error of the gradient method, after k-steps is $\|\vec{c}_k - \vec{c}_*\|_A \leq \frac{(k-1)^k}{(k+1)^k} \|\vec{c} - \vec{c}_*\|_A$