Lecture 5

Abstract Framework for F.E.M.:

Quadratic Minimization Problem:

Find u: $\mathcal{E}(v) \geq \mathcal{E}(u)$, $\forall v$ on a Hilbert space $\mathcal{H}$.

where $\mathcal{E}(v) = \frac{1}{2}a(v,v) - f(v)$

(1st) $a: \mathcal{H}\times\mathcal{H} \to \mathbb{R}$, bilinear form: $a(u+\lambda v, w) = a(u,w) + \lambda a(v,w)$
$a(u, v+\lambda w) = a(u,v) + \lambda a(u,w)$.

(2nd) a is continuous, namely: $a(u,v) \leq C\|u\|\|v\|$

(3rd) a is coercive, namely for $a(u,u) \geq \alpha\|u\|^2$, α

Continuity: $a(u,v) \leq C\|u\|_{H^1}\|v\|_{H^1}$?

$$\int_\Omega \nabla u \cdot \nabla v \, dx \leq \|\nabla u\|_{L^2}\|\nabla v\|_{L^2} \leq \|u\|_{H^1}\|v\|_{H^1}$$

Coercive: $a(u,u) \geq \alpha\|u\|^2_{H^1}$?

$$\int_\Omega |\nabla u|^2 dx = \frac{1}{2}\int_\Omega (\nabla u)^2 dx + \frac{1}{2}\int (\nabla u)^2 dx$$

$$\leq \min\left\{\frac{1}{2}, \frac{\tilde{\alpha}}{2}\right\}\left(\int_\Omega |\nabla u|^2 + \int_\Omega u^2\right)$$

$$= K\|u\|^2_{H^1} \quad \text{where } K = \min\left\{\frac{1}{2}, \frac{\tilde{\alpha}}{2}\right\}$$

In our setting $(f,v)_{L^2} =: f(v)$

Is continuous since, $(f,v)_{L^2} \le \|f\|_{L^2}\|v\|_{L^2}$

$$\le \|f\|_{L^2}\|v\|_{H^1} \quad \text{i.e.} \quad f(v) \le C\|v\|_{H^1}.$$

Gradient test $\Rightarrow$ Minimization problem $\Rightarrow$ weak formulati[on]

Find $u \in H$ such that $\frac{1}{2}\{a(u,v) + a(v,u)\} = f(v)$, $\forall v \in H$.

which makes sense whether a is symmetric or not.

Existence of a minimizers by Lax-Milgram theorem.

~~Existence of minimizer~~

(a) If a is a bilinear form, which is continuous and coercive, and f is linear and continuous, then the weak form admits a unique solution.

(b) If a is symmetric, then the minimization problem admits a unique solution.

We can apply this to two problems:

① $H = H_0^1(\Omega)$: the pde
$$\left.\begin{aligned} -\Delta u &= f \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned}\right\}$$
admits a unique solution (weak)

② $H = H_h$: the FEM. produces a unique solution.

In fact, for ② one can also argue that, from coercivity the matrix A is positive definite.

Two Problems: PDE weak form

Find $u \in H$: $a(u,v) = f(v)$, $\forall v \in H$

FEM: discretized weak form: choose $H_h \subseteq H$.

Find $u_h \in H_h$: $a(u_h, v_h) = f(v_h)$, $\forall v_h \in H_h$.

Cea's Lemma: $\|u - u_h\|_{H'} \leq \frac{C}{\alpha} \inf_{v_h \in H_h} \|u - v_h\|_H$

from continuity

from coercivity

Proof: $a(u,v) = f(v)$, $\forall v \in H$
$a(u_h, v_h) = f(v_h)$, $\forall v_h \in H_h$

$a(u - u_h, v_h) = 0$ (Galerkin orthogonality)

Coercivity: $\alpha \|u - u_h\|_H^2 \leq a(u - u_h, u - u_h) = a(u - u_h, u$
$+ \underbrace{a(u - u_h, v_h - u_h)}_{0}$

Continuity: $\leq C \|u - v_h\|_H \cancel{\|u - u_h\|_H}$

~~The Laplace equation corresponds to a~~

Consider the problem: $-\partial_x(\tilde{a}(x)\, \partial_x u) = f$ in Ω
$u = 0$ on $\partial\Omega$

Dirichlet: The solution u is a solution of the minimizatio problem. Find $u \in H_0^1(\Omega)$: $E(v) \geq E(u)$, $\forall v \in H_0$

where $E(v) = \frac{1}{2} \int_\Omega \tilde{a}(x) (\partial_x v)^2 - \int f v \, dx$

Here $a=(u,v)=\int_{\Omega} \tilde{a}\,(\partial_x u)(\partial_x v)\,dx$

Proof is the same as for $\tilde{a}=1$: $E(u+h)-E(u)$

$\ldots$

n-dimensional case: $\tilde{a}_{ij}$ sufficiently smooth (a_{ij}) ~~pos~~ ~~and~~ sym

$$\left.\begin{array}{l} -\vec{\nabla}\cdot(\bar{A}\nabla u) = f \text{ in } \Omega \\ u=0 \text{ on } \partial\Omega \end{array}\right\} \bar{A} \text{ is symmetric and positive definite}$$

$u \in H_0^1(\Omega)$ which minimizes $E(v)=\frac{1}{2}\int (\bar{A}\nabla u)\cdot\nabla v - \int f v$ over $v \in H_0^1(\Omega)$

Coercivity of bilear form, follows from Poincaré Friedrichs inequality.

Proof of Star Lemma

Note that $\sum_{i,j=1}^{k} a_i(p_i-p_0) = \sum_{i=0}^{k} a_i(p_i-p_0)$

$$= \underbrace{\sum_{i=0}^{k} a_i p_i}_{\leq 0} - p_i \underbrace{\sum_{i=0}^{k} a_i}_{=0} \leq 0$$

Furthermore, $a_i<0$ for all i and $p_i \leq p_0 \rightarrow a_i(p_i-$

$\nexists\ \forall i=1,\ldots,k : a_i<0 \rightarrow p_i<p_0$ unless $p_i=p_0$.