Lister Learning + Teaching centre
Room 3.3

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Last week: $-\partial_x^2 u = f$ in $(0,1)$ $\longleftrightarrow$ minimize $E(v)$

$u(0) = u(1) = 0$

- Weak formulation: $\int_0^1 (\partial_x u)(\partial_x v) = \int_0^1 fv, \ \forall v \in H_0^1(0,1)$
- Discretization: $\int_0^1 (\partial_x u_h)(\partial_x v_h) = \int_0^1 f v_h, \ \forall v_h \in H_h(0,1)$
- Galerkin orthog: $\int_0^1 (\partial_x(u-u_h))(\partial_x v_h) = 0, \ \forall v_h \in H_h$
- Cea's lemma: FE give quasi-optimal solution:

$$\|u-u_h\|_{H^1} \leq C \inf_{v_h \in H_h} \|u - v_h\|_{H^1}.$$

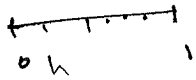
Today:
- Convergence rates: How fast does $u_h \to u$?
- Computable estimates for $u - u_h$
  - $\rightarrow$ local: A number for each triangle.
- Abstract setting: Bilinear forms, coercivity
- How to solve $A\vec{c} = \vec{F}$?
- Convergence of finite differences.

From Cea's lemma: $\|u-u_h\|_{H^1} \le c \inf_{v_h} \|u-v_h\|_{H^1}$

FE error

error of best possible appr. of u by a function in H_h.

<u>Now</u>: $(0,1)$



$H_h = \{$ continuous p.w. linear fcts, kinks at $h, 2h, \dots$ which vanish at $x=0$ and $x=1$ $\}$

[To estimate $\|u-v_h\|_{H^1}$ we need to find one reasonable v_h.]

$\Rightarrow$ <u>Choose:</u> $v_h(x)$ = p.w linear function s.t.
$v_h(jh) = u(jh) \ \forall j.$

$$e_h(x) := u(x) - v_h(x) \quad = 0 \quad \forall x = jh.$$

Use Poincaré - Friedrich inequality on $[jh, (j+1)h]$

$$\int_0^1 e_h^2 = \sum_j \int_{jh}^{(j+1)h} e_h(x)^2 \le \sum_j C_1 \int_{jh}^{(j+1)h} (\partial_x e_h)^2 = h^2 \|\partial_x e_h\|^2_{L^2}$$

easy to see: $((j+1)h - jh)^2 \simeq C_1$
$h^2 \simeq$

Using a variant of P-F ineq: $\int (w-\bar{w})^2 \le C_1 \int \partial_x w$ where $\bar{w} = \frac{1}{|\Omega|}\int w$, this inequality holds

$$\|u-v_h\|^2_{H^1} = \underbrace{\|u-v_h\|^2_{L^2}}_{\le h^2 \|\partial_x(u-v_h)\|^2_{L^2}} + \|\partial_x(u-v_h)\|^2_{L^2}$$

$$\le (1+ch^2) \underbrace{\|\partial_x(u-v_h)\|^2_{L^2}}_{\text{estimate this}}$$

$$\le \sum_j h \|\partial_x^2 (u-\cancel{v_h})\|^2_{L^2(jh,(j+1)h)} = ch^2 \sum_j \|\partial_x^2 u\|^2_{L^2} = h^2 \|\partial_x^2 u\|^2_{L^2}$$

$\partial_x^2 v_h = 0$ because v_h is linear on $(jh, (j+1)h)$

<u>Thm</u>: $\|u-u_h\|_{H^1} \le ch\,\|\partial_x^2 u\|_{L^2}$

To summarize: If $u \in H^2(\Omega)$, then the FE solution u_h on an equidistant mesh will converge linearly:

$\left[H^1\text{-Error} \le ch. \right]$
$\quad \|$
error in energy

Sometimes care about other notions of error!

- Least squares error: $\|u-u_h\|_{L^2} \le ch^2 \|\partial_x^2 u\|_{L^2}$ (one order faster)

- Maximum error: $\|u-u_h\|_{L^\infty} \le C\|u-u_h\|_{H^1} \le ch\|\partial_x^2 u\|$ (possibly faster)

Aubi-Nitsche trick: Let w s.t. $\quad -\partial_x^2 w = e_h = u-u_h$ (solution to dual problem)
$w(0) = w(1) = 0$

$$\|u-u_h\|_{L^2}^2 = (u-u_h, u-u_h)_{L^2} = -(u-u_h, \partial_x^2 w)$$

$$= (\partial_x(u-u_h), \partial_x w) - \underbrace{(\partial_x(u-u_h), \partial_x w_h)}_{=0}$$

Galerkin orthogon.

Choice of w_h: $\quad w_h(jh) = w(jh) \quad \forall j$
$w_h(x) =$ p.w. linear interpolation between the nodes

$$\|u-u_h\|_{L^2}^2 = (\partial_x(u-u_h), \partial_x(w-w_h))$$

$$\overset{(c.s)}{\le} \|\partial_x(u-u_h)\|_{L^2}\, \underbrace{\|\partial_x(w-w_h)\|_{L^2}}_{\le ch\|\partial_x^2 w\|_{L^2}}$$

$\partial_x^2 w = -(u-u_h)$

$$\le \left(h\|\partial_x(u-u_h)\|_{L^2}\right)\|u-u_h\|_{L^2}$$

Young's ineq. $ab \le \frac{a^2}{2} + \frac{b^2}{2}$

$$\le \frac{h^2}{2} \|\partial_x (u-u_h)\|_{L^2}^2 + \underbrace{\frac{1}{2}\|u-u_h\|_{L^2}^2}_{\text{move to L.H.S.}}$$

$$\Rightarrow \frac{1}{2}\|u-u_h\|_{L^2}^2 \le \frac{h^2}{2}\|\partial_x(u-u_h)\|_{L^2}^2$$

$$\le h^4 \|\partial_x u\|_{L^2}^2 \qquad \blacksquare$$

Upshot: FEM converges

$$\|e\|_{H^1} \le ch, \quad \|e\|_{L^2} \le ch^2$$

In 2D the same results hold, but estimating $\|u - v_h\|_{H^1}$ gets a bit more tedious. (Details in Braess, Chp II.6)

Key result:

$$\exists v_h : \|u - v_h\|_{H^1} \le ch \sum_{i,j} \|\partial_{x_i}\partial_{x_j} u\|_{L^2}$$

Question: Can we compute the error $u(x) - u_h(x)$?

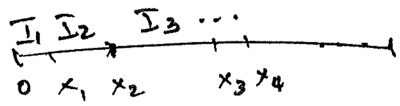
"a posteriori error estimates": Do a computation, see how good it is
where is the error large?

$$\boxed{\begin{aligned} -\partial_x^2 u &= f \\ u(0) = u(1) &= 0 \end{aligned}}$$

Estimate the error in a computable way:

$$\frac{1}{C}\|u-u_h\|^2_{H^1} \le \int_0^1 [\partial_x(u-u_h)]^2$$

P.F. inequality



$$[0,1] = \cup I_j$$

$$I_j = [x_{j-1}, x_j]$$

$$= \int_0^1 \partial_x(u-u_h)\,\partial_x u - \underbrace{\int_0^1 \partial_x(u-u_h)\,\partial_x u_h}_{=0} \quad \forall v_h \in H_h.$$

$\partial_x v_h$ by Galerkin orth.

$$= \int_0^1 \partial_x(u-u_h)\,\partial_x(u-v_h)$$

$$= \int_0^1 \partial_x u\, \partial_x(u-v_h) - \int_0^1 \partial_x u_h\, \partial_x(u-v_h)$$

weak formulation $= \int_0^1 f(u-v_h)$

$$= \sum_j \int_{I_j} \partial_x u_h\, \partial_x(u-v_h)$$

integrate by parts

$$= \sum_j \int_{I_j} -\partial_x^2 u_h (u-v_h) + [\partial_x u_h(x_j^+) - \partial_x u_h(x_j^-)]\cdot[u(x_j)-v_h(x_j)]$$

jumps

$$= \underbrace{\int_0^1 (f+\partial_x^2 u_h)(u-v_h)}_{=0 \text{ because the PDE (for the exact solution)}}$$

$$+\sum_j (\text{jump of } \partial_x u_h \text{ at } x_j)\cdot(u(x_j)-v_h(x_j))$$

Note: For exact solution both terms are $=0$.

choose v_h s.t. $v_h(x_j) = u(x_j)$

$$\overset{C.S.}{\le} \sum_{I_j} \|f+\partial_x^2 u_h\|_{L^2(I_j)} \|u-v_h\|_{L^2(I_j)} + \sum_j (\text{jump of } \partial_x u_h \text{ at } x_j)\,\|u-v_h\|_{H^1}$$

Note: $H^1(0,1) \hookrightarrow C^0(0,1)$, i.e., $|w(x)| \le C_1 \|w\|_{H^1}$ $\forall w \in H^1$

$$\le \tilde{C}_1 \sum_j \|f+\partial_x^2 u_h\|_{L^2(I_j)} \underbrace{h\,\|\partial_x u\|_{L^2(I_j)}}_{\text{see proof of convergence rates}} + \sum(\text{jumps})\underbrace{\|u-v_h\|}_{\le C}$$

$$\le \tilde{\tilde{C}} \sum_j h\|f+\partial_x^2 u_h\|_{L^2} + \sum \text{jumps} \Rightarrow \text{computable}$$

We have seen a computable error estimate

$\|u-u_h\|_{H^1} \le C\sum_j \eta_{I_j}^2$, where η_{I_j} is computable from u_h in I_j.

Adaptive FEM:

Start: coarse mesh, solve $-\partial_x^2 u = f$ on this mesh

~~while $\sum \eta_{I_j}^2 \ge$ tolerance~~

compute η_{I_j}

If $\sum \eta_{I_j}^2 <$ tolerance $\to$ stop (solution good enough)

else
mark I_j where η_{I_j} is big
divide marked I_j's into smaller intervals/Δs
then solve $-\partial_x^2 u = f$ on new mesh