 23/01/2017

Last week : Finite elements for Laplace Equation

PDE: $\begin{cases} -\partial_x^2 u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$ $\Rightarrow$ Find $u \in H_0^1(\Omega)$

such that, for all $v \in H_0^1(\Omega)$, $E(u) \leq E(v)$ where

$$E(v) = \frac{1}{2}\int_\Omega (\nabla v)^2 dx - \int_\Omega f v \, dx$$

Here $H_0^1(\Omega) = \{ v \in L^2(\Omega) : \nabla v \in L^2(\Omega) \}$

endowed with the scalar product :

$$(v,w)_{H^1} = \int_\Omega (vw + \nabla v \cdot \nabla w)\, dx$$ and norm

$$\|v\|_{H^1} = (v,v)_{H^1}^{1/2}.$$

Finite Elements: Choose a subspace $H_h \subseteq H_0^1(\Omega)$ of finite dimension.

FEM problem: Find $u_h \in H_h$ such that for all $v_h \in H_h$

$$E(u_h) \leq E(v_h)$$

Fix basis: $\{\Lambda_1(x), \dots \Lambda_N(x)\}$ of H_h

$$u_h(x) = \sum_{j=1}^{N} c_j \Lambda_j(x)$$

$$\partial_x \Lambda_j(x) = \begin{cases} \dfrac{1}{x_j - x_{j-1}} & \text{if } x \in (x_{j-1}, x_j) \\[2ex] \dfrac{-1}{x_{j+1} - x_j} & \text{if } x \in (x_j, x_{j+1}) \end{cases}$$

$$A_{ij} = \int_\Omega \nabla\Lambda_i \cdot \nabla\Lambda_j \, dx \quad , \quad F_j = \int_\Omega f \Lambda_j \, dx$$

$$\boxed{A\vec{C} = \vec{F}}$$

Finite Element Primer: Calculate everything in detail for $0, h, 2h, \dots, 1$

Result: $A = \begin{bmatrix} 2 & -1 & & 0 \\ -1 & 2 & -1 & \\ & -1 & 2 & -1 & 0 \\ 0 & & \ddots & \ddots & \ddots \\ & & & -1 & 2 & -1 \end{bmatrix}$ 2's are on the diagonal $i=j$, -1's are for $i = j \pm 1$

Questions: Does the FEM equation $A\vec{C} = \vec{F}$ have a unique solution? $\Longleftrightarrow$ Is there a unique minimizer of the energy E on H_h?

Lemma: A is positive definite. show $\forall \vec{x} \in \mathbb{R}^N \setminus \{0\}$ $x^T A x > 0$

Proof: Using Poincaré Friedrichs inequality

$$\int_\Omega |\nabla u|^2 dx \geq C \int_\Omega |u|^2 dx \quad \text{and } u \in H_0^1(\Omega)$$

$$v_h = \sum_j x_j \Lambda_j \in H_h$$

Continue Proof of Lemma :

$$U_h = \sum_j x_j \Lambda_j \in H_h = \sum_{ij} x_i A_{ij} x_j = \cancel{\sum_{ij} x_i x_j \int_\Omega \nabla\Lambda_i \cdot \nabla\Lambda_j}$$

$$= \sum_{ij} x_j x_i \int_\Omega (\vec{\nabla}\Lambda_i)\cdot(\vec{\nabla}\Lambda_j) \qquad \int_0^1 (\partial_x u)^2 dx \geq \int_0^1 |u|^2 dx$$

$$= \int_\Omega \vec{\nabla}\left(\sum_i x_i \Lambda_i\right)\cdot\vec{\nabla}\left(\sum_j x_j \Lambda_j\right)$$

$$= \int_\Omega (\vec{\nabla} V_h)\cdot(\vec{\nabla} V_h) = \int_\Omega (\nabla V_h)^2 \geq \int_\Omega (V_h)^2 > 0$$

Thus, $A_{ij} = A_{ji}$, i.e. A is symmetric.

In particular, $A\vec{c} = \vec{F}$: has a unique solution

A symmetric

<u>Theorem</u> : $\|u - u_h\|_{H^1}^2 = \|u - u_h\|_{L^2(\Omega)}^2 + \|\vec{\nabla}u - \vec{\nabla}u_h\|_{L^2(\Omega)}^2$

$\leq C$, $\forall V_h \in H_h$

independent of H_h

<u>Proof</u> : Gradient tests for u and u_h = weak formulation of PDE.

$$\int_\Omega (\vec{\nabla}u)\cdot(\vec{\nabla}v)\, dx = \int_\Omega f v\, dx \quad \forall v \in H_0^1$$

$A\vec{c} = \vec{F} \Leftrightarrow$

$$\int_\Omega (\vec{\nabla}u_h)\cdot(\vec{\nabla}V_h) = \int_\Omega f V_h , \forall V_h \in H_h$$

Taking the difference: $\int_\Omega \vec\nabla(u-u_h)\cdot\vec\nabla v_h = \int_\Omega f(v_h - v_h) = 0$

we are getting as close as possible to u in H_h as possible

Galerkin orthogonality $(u-u_h) \perp H_h$

$$\|\vec\nabla(u_h-u)\|^2_{L^2(\Omega)} = \int_\Omega (\vec\nabla(u_h-u))^2 dx$$

$$= \underbrace{\int_\Omega \vec\nabla(u_h-u)\cdot\vec\nabla u_h \, dx}_{0} - \int_\Omega \vec\nabla(u_h-u)\cdot\vec\nabla u \, dx$$

$$= \int_\Omega \vec\nabla(u_h-u)\cdot\vec\nabla v_h \, dx - \int_\Omega \vec\nabla(u_h-u)\cdot\nabla u \, dx$$

$$= \int_\Omega \vec\nabla(u_h-u)\cdot\vec\nabla(v_h-u)\, dx \le \left(\int_\Omega (\vec\nabla(u_h-u))^2\right)^{1/2} \left(\int_\Omega \vec\nabla(v_h-u)^2\right)^{1/2}$$

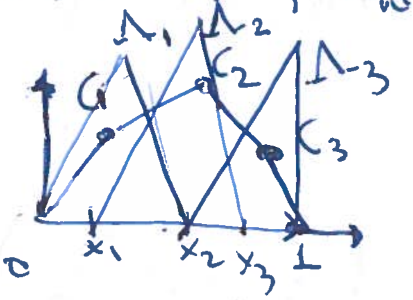
Cauchy-Schwartz

$$\Leftrightarrow \|\vec\nabla(u-u_h)\|_{L^2(\Omega)} \le \|\vec\nabla(u-v_h)\|^2_{L(\Omega)}$$

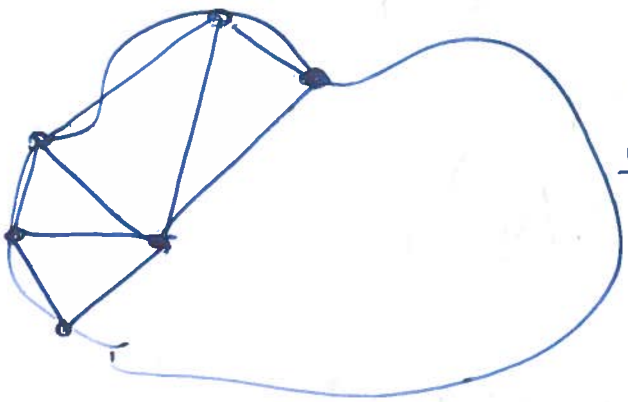
Also $\|u-u_h\|^2_{L^2} \le \frac{1}{c}\|\vec\nabla(u-u_h)\|^2_{L^2(\Omega)} \le \frac{1}{c}\|\vec\nabla(u-v_h)\|^2_{L^2}$

To solve PDE $\left.\begin{array}{ll} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{array}\right\}$ numerically

Solve $A\vec{c} = \vec{F}$, $u_h(x) = \sum_j c_j \Lambda_j(x)$



1^{st} step: Choose the discretisation. For today (linear finite elements) Piecewise linear functions which are continuous ($\Leftarrow$ no jumps).



Ω

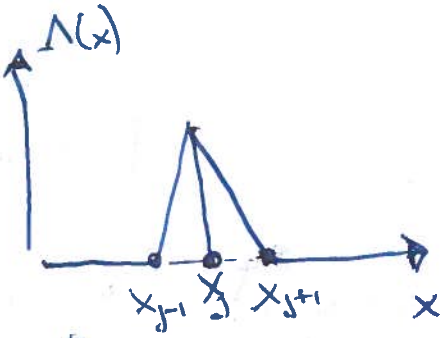
triangulation $\Omega_h = \bigcup_j T_j$

thus $\Omega = \Omega_h$
or not.

2nd step: Basis of H_h = Hat functions: 1d 

2d 

$\vec{\nabla} \Lambda(x) = ?$



$$\partial_x \Lambda(x) = \begin{cases} \frac{1}{x_j - x_{j-1}}, & x \in (x_{j-1}, x_j) \\ \frac{-1}{x_{j+1} - x_j}, & x \in (x_j, x_{j+1}) \\ 0, & \text{else} \end{cases}$$

3rd step   Main algorithmic part:

Calculate $A_{ij} = \int_{\Omega} (\vec{\nabla}\Lambda_i)\cdot(\vec{\nabla}\Lambda_j)$ using the formulas from 2nd

$$\int_{\Omega} (\vec{\nabla}\Lambda_i)\cdot(\vec{\nabla}\Lambda_j) = \sum_{\substack{T \text{ triangles} \\ \text{in } \Omega_h}} \int_T (\vec{\nabla}\Lambda_i \cdot \vec{\nabla}\Lambda_j)$$

Note: because H_h consists of piecewise linear functions
$\vec{\nabla}\Lambda$ is constant in every T.

$$A_{ij} = \sum_T (\vec{\nabla}\Lambda_i)_T \cdot (\vec{\nabla}\Lambda_j)_T$$

Efficiently: Compute numbers for all T, and then correct numbers from triangles that contain nodes i and j.

4th step   Same for $\vec{F}$: $F_j = \int_{\Omega} f(x)\Lambda_j(x)\,dx$

need some way of numerically computing $\int_{\Omega}$.

5th step   Solve $A\vec{c} = \vec{F}$

6th step   Post processing: Plot solution, calculate physically relevant information