Recall from last week: 16.1.2018

Solving PDE on the computer: • How?
• Does it work?
• Is it efficient?

Finite differences: Based on difference quotients
For example $u_x(x) \approx \frac{u(x+h)-u(x)}{h}$ for fixed h.

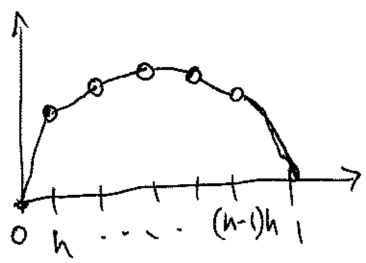
Straightforward to generalize to higher D.

Given a numerical method (FD, FE) for a PDE ($\partial_x^2 u = f$), does it converge? In which sense?

Finite Elements

- Uses setup via minimizers of Energy.
- Choose finite dim subspace $H_h \subseteq H_0^1(\Omega)$ and minimize
$$E(v) = \frac{1}{2}\int_\Omega |\nabla v|^2 dx - \int_\Omega fv,$$
over H_h.
- Numerical soln. is given as $u_h \in H_h$ s.t. $E(v_h) \geq E(u_h)$ $\forall v_h \in H_h$.

More concretely: $\partial_x^2 u = f$ on $[0,1]$; $u : [0,1] \to \mathbb{R} \in H_0^1(\Omega)$
$u(0) = u(1) = 0$. $u_h : [0,1] \to \mathbb{R}$



And consider piecewise linear approx. in each interval.

The Space H_h of these piecewise linear functions is finite dimensional.

$\dim H_h = \#$ interior nodes $\approx \frac{1}{h}$

Write down a basis of H_h:

$$\Lambda_j(x) = \begin{cases} 0 & x \notin [x_{j-1}, x_{j+1}] \\ 1 & x = x_j \\ 0 & x = x_{j-1} \text{ or } x_{j+1} \\ \text{linear in} & [x_{j-1}, x_{j+1}]. \end{cases}$$

$$u_h(x) = \sum_j c_j \Lambda_j(x).$$

The finite dimensional problem reduces to liner algebra.

That is: Minimize $E(v_h)$, where $v_h(x) = \sum_j c_j \Lambda_j(x)$

$$\begin{aligned} E(v_h) &= E\Big(\sum_j c_j \Lambda_j(x)\Big) = \frac{1}{2}\int_\Omega \Big[\nabla \sum_j c_j \Lambda_j(x)\Big]\Big[\nabla \sum_k c_k \Lambda_k(x)\Big] - \int_\Omega f \sum_j c_j \Lambda_j(x) \\ &= \frac{1}{2}\sum_{j,k} c_j c_k \underbrace{\int_\Omega \nabla\Lambda_j(x)\cdot\nabla\Lambda_k(x)}_{=:A_{jk}} - \sum_j c_j \underbrace{\int_\Omega f(x)\Lambda_j(x)}_{F_j} \end{aligned}$$

$$\big(E(\vec{c})\big) = \frac{1}{2}\vec{c}^T A \vec{c} - F\vec{c}$$

So, $\min_{\vec{c}\in\mathbb{R}^N} E(\vec{c})$ is quadratic minimization problem.

Solve by derivative test: $\frac{\partial}{\partial c_j} E(\vec{c}) = 0, \ \forall j.$

$\rightsquigarrow (Ac - F)_j = 0, \ \forall j. \quad \rightsquigarrow \quad Ac = F$, Lin. algebra

In 1D matrix A same as in finite difference case.

What is the derivative of a nondiff. function?

Key idea: Integration by parts

<u>Def:</u> Let $u \in L^2(\Omega)$. Then define $\partial_x u$, if it exists, as a function $v \in L^2(\Omega)$ s.t. "$\int_\Omega (\partial_{x_i} u \cdot \phi) = -\int_\Omega u \cdot \partial_{x_i} \phi + b.c.$"

$$(1) \quad \int_\Omega v\phi = -\int_\Omega u \partial_{x_i} \phi, \quad \forall \phi \in C^\infty(\Omega)$$

0 on $\partial\Omega$.

Notation: $\partial_{x_i} u := v$.

<u>Remark:</u>
Any v_1, v_2 which satisfy (1) differ only on set of measure 0.

<u>Ex:</u> $u(x) = |x|$

$$\partial_x u = \begin{cases} -1 & x<0 \\ \text{"anything"} & x=0 \\ 1 & x>0 \end{cases}$$

<u>Def:</u> (Sobolev) $u \in H^k(\Omega)$, $k \in \mathbb{N}_0$ if the weak derivatives up to order k exist and $\int_\Omega |\partial^\alpha u|^2 dx < \infty$.

for multiindex α.

Norm: $\|u\|_{H^k(\Omega)} := \left\{ u \in L^2 : \left(\int_\Omega |u|^2 + \int_\Omega |\nabla u|^2 + \dots + \int_\Omega |\nabla^k u|^2 \right)^{\frac{1}{2}} < \infty \right\}$.

<u>Thm:</u> (Poincaré-Friedrich ineq.)

$$\int_0^1 w^2 \leq \int_0^1 |\partial_x w|^2 \qquad \forall w \in H_0^1(\Omega)$$

<u>Proof:</u> (for $w \in C^1$) ← General case follows as C^1 dense in H^1.

By FTC $\quad w(x) = w(0) + \int_0^x \partial_x w = \int_0^x \partial_x w.$

$$w^2(x) = \left(\int_0^x \partial_x w\right)^2 \overset{C.S.}{\leq} \left[\left(\int_0^x 1^2\right)^{\frac{1}{2}} \left(\int_0^x (\partial_x w)^2\right)^{\frac{1}{2}}\right]^2 \leq \int_0^1 (\partial_x w)^2.$$

Thm: Let $u \in H_0^1(\Omega)$ be the minimizer of E over $H_0^1(\Omega)$ and $u_h \in H_h$ be the minimizer of E over H_h.

Then $\|u-u_h\|_{H_0^1(\Omega)} \leq C \|u-v_h\|_{H^1(\Omega)}$, $\forall v_h \in H_h$.

indept. of h.

- This means that up to a constant FE solution gives us the best possible approximation of u among all functions in H_h.