What is NA?

Framework for FEM:
- H – Hilbert space ($H^1(\Omega)$ or $H^1_0(\Omega)$ for Δ)
- bilinear form $a: H\times H \to \mathbb{R}$, continuous and coercive.

Solve weak formulation of PDE:

$$\text{Find } u\in H: \quad a(u,v)=f(v) \quad, \forall v\in H.$$

Discretization $H_h < H$.

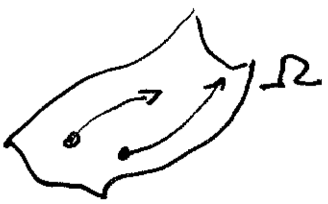
$$\text{Find } u_h\in H_h: \quad a(u_h,v_h)=f(v_h), \quad \forall v_h\in H_h.$$

- Cea's lemma $\Rightarrow$ convergence.

Today: Coercivity doesn't hold.

Ex.:
- Stokes problem
- Mixed formulation.

Navier Stokes:



$u: \Omega \to \mathbb{R}^n$ velocity

$p: \Omega \to \mathbb{R}$ pressure

incompressible:
$$\partial_t u + (u\cdot\nabla)u - \nu\Delta u - \nabla p = f$$
$$\nabla\cdot u = 0$$
+ I.C. + B.C.

$\downarrow$

Stokes problem:
$$-\Delta u - \nabla p = f$$
$$\nabla\cdot u = 0$$
+ B.C.

$$\int_\Omega \sum_{i,j=1}^{n} \partial_j u_i \, \partial_j v_i = \int_\Omega \nabla u : \nabla v$$

Weak formulation:

$$a(\cancel{\nabla} u, \cancel{\nabla} v) - (\nabla p, v) = f(v) \qquad v : \Omega \to \mathbb{R}^n$$

$$(\nabla \cdot u, q) = 0 \qquad q : \Omega \to \mathbb{R}$$

$$a(u,v) + \overbrace{(p, \nabla \cdot v)}^{= b(v,p)} = f(v)$$

Find $(u,p) \in H_0^1(\Omega)^n \times L_0^2(\Omega)$:

$$a(u,v) + b(v,p) = f(v) \qquad \forall v \in H_0^1(\Omega)^n$$

$$b(u,q) = 0 \qquad \forall q \in L_0^2(\Omega)$$

$$\int_\Omega p = 0$$

Problem: $a(\cdot,\cdot)$ is coercive bilinear form, but we have

$$\begin{pmatrix} A & B^T \\ B & 0 \end{pmatrix} \begin{pmatrix} u \\ p \end{pmatrix} = \begin{pmatrix} F \\ 0 \end{pmatrix}$$

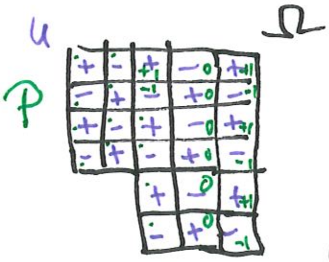
$A > 0$. ✓ However, not true that $\begin{pmatrix} A & B^T \\ B & 0 \end{pmatrix} > 0$.

As long as $\begin{pmatrix} A & B^T \\ B & 0 \end{pmatrix}$ is invertible, the original problem is still wellposed. $\exists!$ soln. (u,p).

Is $\begin{pmatrix} A & B^T \\ B & 0 \end{pmatrix}_h$ invertible? No. $\begin{pmatrix} A & B^T \\ B & 0 \end{pmatrix}_h$ may have small eigenvalues of size h or even be singular.

$\Rightarrow$ Num. methods easily blow up.

Choose $n=2$: $X_h \subseteq H_0^1(\Omega)^2$, $X_h = \{ v \in C^0(\bar{\Omega})^2 : v|_{\Box} = a + bx + cy + dxy \}$ (3.)



$M_h \subseteq L_0^2(\Omega)$, $M_h = \{ q \in L_0^2(\Omega) : q|_{\Box} = \text{constant} \}$

Oscillations become stronger as mesh is refined.

<u>Ex.:</u> Laplace eq. in mixed formulation:

$$-\Delta u = f$$
$$-\nabla \cdot (\nabla u) = f$$

In many applications one wants to also solve for $(u, \nabla u)$.

Let $\sigma = \nabla u$. Then, $\begin{cases} -\nabla u + \sigma = 0 \\ \nabla \cdot \sigma = -f \end{cases}$

Weak form: Find $(\sigma, u) \in (L^2(\Omega))^n \times H_0^1(\Omega)$:

PRIMAL
$$\begin{cases} a(\sigma,\tau) + b(\tau,u) = 0 \\ b(\sigma, v) = f(v) \end{cases} \quad \begin{cases} \int_\Omega \sigma \tau - \int_\Omega \tau \cdot \nabla u = 0 & \forall \tau \in L^2(\Omega)^n \\ -\int_\Omega \sigma \cdot \nabla v = -\int_\Omega f v & \forall v \in H_0^1(\Omega) \end{cases}$$

Find $(\sigma, u) \in H(div) \times L^2(\Omega)$.

DUAL
$$\begin{cases} a(\sigma,\tau) + \tilde{b}(\tau,u) = 0 \\ \tilde{b}(\sigma, v) = f(v) \end{cases} ; \quad \tilde{b}(\tau, u) = \int_\Omega (\operatorname{div} \tau) u$$

$$H(div) = \{ \tau \in L^2(\Omega)^n : \operatorname{div}(\tau) \in L^2(\Omega) \} \subseteq H^1(\Omega).$$

The system $\begin{pmatrix} A & B^T \\ B & 0 \end{pmatrix}\begin{pmatrix} u \\ \lambda \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix}$ is equivalent

to "Minimize $E(u) = \frac{1}{2}u^T A u - f^T u$
subject to $Bu = g$ ".

We want generalization of coercivity which guarantees invertibility of $\begin{pmatrix} A & B^T \\ B & 0 \end{pmatrix}$.

Thm: (Babuska: Corollary of closed range thm.)

U, V - Hilbert. $L : U \to V'$ linear.
then L is isomorphism iff the bilinear form of $\ell(u,v) := \langle Lu, v \rangle$ satisfies:

(1) Continuity: $|\ell(u,v)| \leq C \|u\|_U \|v\|_V$.

(2) infsup condition (LBB): $\exists \alpha > 0 : \sup_{v \in V} \frac{\ell(u,v)}{\|v\|_V} \geq \alpha \|u\|_U$:

Remark: $a(\cdot,\cdot) : H \times H \to \mathbb{R}$ bilinear & coercive: $a(u,u) \geq \alpha \|u\|^2$
$\Rightarrow$ infsup

(3) $\forall v \in V, \exists u \in U : \ell(u,v) \neq 0$.

Proof: Corollary of Closed range thm. p. 125 Braess.

<u>Corollary:</u> (Cea's Lemma) Suppose $\ell(\cdot,\cdot)$ satisfies conditions (1), (2), (3) and suppose U_h, V_h subspaces of U, V:

(!) (2), (3) also hold for U_h, V_h.

Then the solns. $u \in U$, $u_h \in U_h$ of

$$\ell(u,v) = f(v) \quad , \quad v \in V$$
$$\ell(u_h, v_h) = f(v_h) \quad , \quad v_h \in V_h.$$

satisfy $\|u-u_h\|_U \leq \left(\frac{C}{\alpha}+1\right) \inf_{w_h \in U_h} \|u-w_h\|_U.$

(!) Need to be careful with discretizations.

<u>Proof:</u> By GO $\ell(u-u_h, v_h) = 0$, $\forall v_h \in V_h$.

So, $\ell(u-w_h, v_h) = \ell(u_h - w_h, v_h)$.

Set $f(v) := \ell(u-w_h, v_h) \Rightarrow |f(v)| \leq C\|u-w_h\|\|v\|$

By assumption L_h^{-1} cont. $\Rightarrow \|L_h^{-1}\| \leq \frac{1}{\alpha}$

$\Rightarrow \|u_h - w_h\| \leq \frac{1}{\alpha}\|f\| \leq \frac{C}{\alpha}\|u-w_h\| \quad \forall w_h \in U_h.$ #

For $L = \begin{pmatrix} A & B^T \\ B & 0 \end{pmatrix}$ the thm. can be split on conditions on $a(\cdot,\cdot)$, $b(\cdot,\cdot)$ separately.

Thm: (Brezzi's splitting thm.)

$\begin{pmatrix} A & B^T \\ B & 0 \end{pmatrix}$ is an isomorphism.

$\Longleftrightarrow \begin{cases} \text{(i)} \quad a(u,v) \text{ is coercive; } a(u,u) \geq \alpha \|u\|^2, \forall u \in V \\ \text{where } V = \{v \in X : b(v,\mu) = 0, \forall \mu \in M\}. \\ \text{(ii)} \quad b(v,\lambda) \text{ satisfies inf-sup condition.} \end{cases}$

- (i) usualy clear, (ii) highly nontrivial.