Heiko: Numerical Analysis.

Contents of the course: • Solving PDE (on computer)

⇒ How? – Finite element methods
- Finite difference methods
- Time-stepping schemes
- Numerical linear algebra

(…)

⇒ Do these methods work?
- How to use theorems/analysis to improve them

In this course we are going to study model problems:

<u>Laplace</u> $\Delta u = f$ where $\Delta u = \dfrac{\partial^2 u}{\partial x_1^2} + \dfrac{\partial^2 u}{\partial x_2^2} + \dots + \dfrac{\partial^2 u}{\partial x_n^2}$ in $\mathbb{R}^n$

<u>Heat</u> $\partial_t u - \Delta u = f$

Wave $\partial_t^2 u - \Delta u = f$ and their nonlinear variants.

Dirichlet. b.c.

<u>Today</u>: $\partial_x^2 u = f \; (*)$ in $[0, L=1]$ with $u(0) = u(L) = 0$

→ Given f, find u.

Most naive approach: Analysis text book.

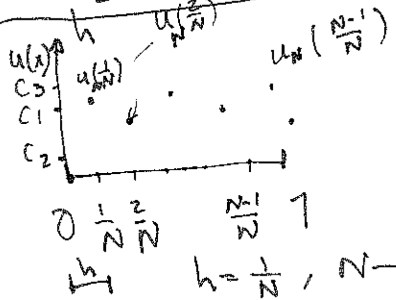
Provided u is differentiable twice

$$\begin{cases} u'(x) = \lim\limits_{h\to 0} \dfrac{u(x+h)-u(x)}{h} \\ u''(x) = (u')'(x) \\ \quad = \lim\limits_{h\to 0} \dfrac{u(x+h)-2u(x)+u(x-h)}{h^2} \end{cases}$$

Fix h small ⇒ $u''(x) \simeq \dfrac{u(x+h)-2u(x)+u(x-h)}{h^2}$

$(*) \quad = f(x)$

$c_j = u_N\left(\frac{j}{N}\right)$



Decompose $[0,1] = \bigcup_{j=0}^{N-1} \left[\frac{j}{N}, \frac{j+1}{N}\right]$

$h = \frac{1}{N}$, $N \to \infty$

Hope: (sometimes true) $u_N(\frac{1}{N}) \to u$

Let's see how to compute u_N: Use $\square$

$$u''(h) \simeq \frac{u(2h) - 2u(h) + u(0)}{h^2} \simeq \frac{c_2 - 2c_1 + c_0}{h^2}$$

and because $u(0) = u(1) = 0 \Rightarrow c_0 = c_N = 0 \Rightarrow u''(h) \simeq \frac{c_2 - 2c_1}{h^2}$

$$u''(2h) \simeq \frac{u(3h) - 2u(2h) + u(h)}{h^2} \simeq \frac{c_3 - 2c_2 + c_1}{h^2}$$

$\vdots$

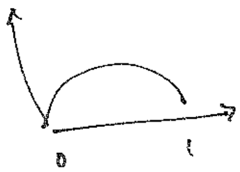
In matrix notation

$$u''(jh) \simeq \frac{c_{j+1} - 2c_j + c_{j-1}}{h^2} \overset{(*)}{=} f(jh) \Rightarrow \frac{1}{h^2}\begin{bmatrix} -2 & 1 & \cdots & 0 \\ 1 & -2 & 1 & \vdots \\ & \ddots & \ddots & 1 \\ 0 & \cdots & 1 & -2 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_{N-1} \end{bmatrix} = \begin{bmatrix} f(h) \\ f(2h) \\ \vdots \\ f((N-1)h) \end{bmatrix}$$

PDE $\begin{cases} \partial_x^2 u = f & \text{in } [0,1] \\ u(0) = u(L) = 0 \end{cases}$ $\leadsto$ (Linear) Algebra $[\quad]\vec{c} = \vec{F}$

Method above: "finite difference method"

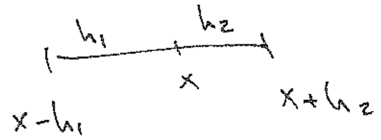
(advantage: easy in easy geometries + for smooth solutions)

1D 

$$u(x) = \frac{1}{2}x(x-1)$$

Finite difference (unequal discret.)

$$\partial^2 u \simeq \frac{2u(x+h_2)}{h_2(h_1+h_2)} - \frac{2u(x)}{h_1 h_2} + \frac{2u(x-h_1)}{h_1(h_1+h_2)}$$



Basic problem: What if the solution is not C^2?

case of re-entrant corner

Second Part : Obtain the same linear algebra problem

for $\partial_x^2 u = -1$ from different point of view: $\boxed{\text{FEM}}$

$u(0) = u(1) = 0$

Physically : Nature does not care about PDES, but it minimises energy : Find "equilibrium" of a physical system.

$\Rightarrow$ Numerical method: Discretise physical system $\to$ minimise discrete energy.

Thm : $\Omega \subseteq \mathbb{R}^n$ bounded, open set. Define $E(u) = \frac{1}{2}\int_\Omega (\nabla u)^2 - \int_\Omega fu$

for suitably nice u (all u for which this makes sense $\to H^1(\Omega)$)

Then the following are equivalent:

(1) u solves $-\Delta u = f$ in Ω
$u = 0$ on $\partial\Omega$

(2) u minimises $E(u)$ ~~over the~~ among all functions of finite energy which vanish on $\partial\Omega$.

Notation: $H_0^1(\Omega) = \left\{ v : \Omega \to \mathbb{R} : \int_\Omega v^2 < \infty, \int_\Omega (\nabla v)^2 < \infty, v = 0 \text{ on } \partial\Omega \right\}$
measurable

Proof : $\forall h \in H_0^1(\Omega)$: $E(u+h) - E(u) =$

$$= \frac{1}{2}\int_\Omega (\nabla(u+h))^2 - \int_\Omega f(u+h) - \frac{1}{2}\int_\Omega (\nabla u)^2 + \int_\Omega fu$$

$$= \int_\Omega \nabla u \cdot \nabla h + \frac{1}{2}\int_\Omega (\nabla h)^2 - \int_\Omega fh$$

integr. by parts

$$= -\int_\Omega (\Delta u)h + \int_{\partial\Omega} \frac{\partial u}{\partial n} \cdot h + \frac{1}{2}\int_\Omega (\nabla h)^2 =$$

$h = 0$ on $\partial\Omega$

$$\Rightarrow E(u+h)-E(u) = -\int_\Omega (\Delta u+f)h + \frac{1}{2}\int_\Omega (\nabla h)^2$$

① $\Rightarrow$ ② : $E(u+h)-E(u) = \frac{1}{2}\int (\nabla h)^2 \geq 0 \quad \forall h \in H_0^1(\Omega)$

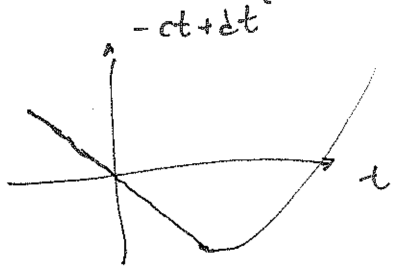
we conclude that $E(u)$ is really the minimum energy.

$$\Rightarrow \forall v \in H_0^1(\Omega) \Rightarrow E(v) \geq E(u)$$

② $\Rightarrow$ ① : $E(u+th)-E(u) \qquad \forall t \in \mathbb{R}, \forall h \in H_0^1(\Omega)$

$$= -t\int_\Omega (\Delta u+f)h + \frac{t^2}{2}\int_\Omega (\nabla h)^2 = -ct + dt^2 \geq 0$$

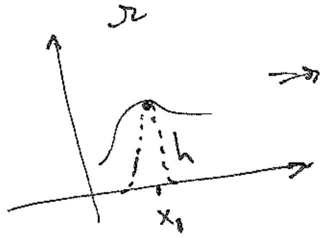
by assumpt. ②



$\Rightarrow 0 = c$ (the only way that ≥ 0 is for $c=0$)

$$\Rightarrow c = \int_\Omega (\Delta u+f)h$$

Assume $\Delta u+f \neq 0$ in x_0



$\Rightarrow \int (\Delta u+f)h > 0$ ↯

<u>Conclusion</u>: $\Delta u+f=0 \Rightarrow$ ① is satisfied ☒.